

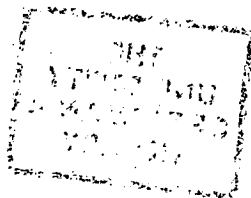
THE GAUGE TECHNIQUE

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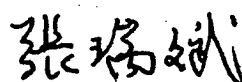
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Zhang Ruibin

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DEDICATED TO
MY WIFE, ZHONGQIU

ABSTRACT

The gauge technique is a nonperturbative scheme for solving the field equations in gauge theories. This thesis is devoted towards understanding and extending the approximation procedures in the gauge technique. In the past the lowest order approximation consisted in only taking into account the longitudinal amplitudes; we have succeeded in generalizing it to incorporate transverse corrections for arbitrary covariant gauges in the context of electrodynamics. As a consequence the nonperturbative results are exact to order e^4 , and the gauge invariance of the vacuum polarization is correspondingly restored by the refined technique.

Contrary to common belief, we have also found that the radiative corrections to the gluon propagator lie solely in the contributions of transverse vertices. Thus in principle the lowest order gauge technique is not readily applicable to the gluon sector of quantum chromodynamics.

Finally we have shown that in two-dimensional electrodynamics (massless Schwinger model) the gauge technique produces exact results because one is able to solve the vector and axial Ward-Takahashi identities uniquely. In this way it is possible to obtain the complete solution for any linear gauge at any temperature.

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I. INTRODUCTION

I.1 INTRODUCTORY REMARKS

Gauge field theories based on the principle of gauge symmetry [1] fill an extremely important position in physics. For three decades their progress has dominated the development of quantum field theory and governed the trends of high energy physics. Even though the status of the so-called grand unified theory, such as the $SU(5)$ and the $SU(4) \times SU(4)$ models [2], is uncertain, and even though there exist stubborn problems in constructing quantum theories for gravity [3], gauge theories have provided us with a systematic method to describe, understand and explore fundamental interactions in the nature.

The $SU(2) \times SU(1)$ gauge theory [4] in combination with the fundamental idea of spontaneous symmetry breaking [5] has been completely verified with the identification of the intermediate vector bosons [6]; not only does it permit all the phenomenological weak models, such as the Fermi V-A interaction, to be cast into a self-consistent theoretical framework, but more importantly it unifies the two different kinds of interactions, namely, the electromagnetic and weak interactions, whose strengths are ostensibly different by an order of 10^5 in magnitude. Quantum chromodynamics [7], the gauge theory of the $SU(3)$ colour group, has also been generally accepted as the appropriate theory for describing the strong interaction: it provides the very natural theoretical foundation for the phenomenological quark model [8] for hadrons and also explains convincingly the experimental results of deep inelastic scattering [9] etc. corresponding to asymptotic freedom [10], a property only exhibited by nonabelian gauge theories in four dimensions.

However all these remarkable properties of gauge theories do not exhaust the range of possibilities. In exploring gauge theories we are limited by the analytical tools presently available, and some aspects of gauge theories remain inaccessible by the nature of these tools. The conventional method (and also the only truly sophisticated approach to the study of quantum field theories) is perturbation theory, where physical quantities are approximated by power expansions in coupling constants through Feynman diagrams. In electroweak theory as well as in the high energy region of strong interactions, where the effective coupling constants are much smaller than unity, one can reasonably hope that when the desired quantities are expanded using Feynman diagrams, the resulting power series in the coupling constants will be convergent, or at least provide a good asymptotic approximation; but in intermediate and low energy strong interactions the effective coupling constant is of the magnitude 1-10 or even large and perturbation theory becomes inappropriate. Some problems are even intrinsically non-perturbative, such as colour confinement, bound states, dynamical symmetry breaking, instantons and solitons, etc. To investigate these problems, people have developed various non-perturbative approximation schemes, among which the popular ones are: renormalization group [11], $1/N$ expansion [12], lattice theories [13] and schemes inspired by the BCS superconductivity theory [14], for example, the Baker-Johnson-Willey program [15,16] and the gauge technique [17,18]. The gauge technique will be the subject of this thesis.

As is well-known, a quantum field theory is completely determined by its Green functions, which are the vacuum expectation values of time ordered products of field operators. Apart from a few solvable lower

dimensional models, such as the Schwinger model [19], the Thirring model [20] etc., the renormalized versions of these functions are not expressible in closed forms in the case of interacting fields. Even perturbatively they can only be studied to the first few orders, higher order terms becoming intractable due to the immense complexity of Feynman integrals. Our limited knowledge is far from enough to determine any Green functions exactly, but fortunately, some of their general properties which are independent of particular theoretical models can be obtained by formal considerations of quantum field theory. In the following we will introduce three of them, namely, the Dyson-Schwinger equations, the Ward-Takahashi identities and the Källén-Lehmann-Wightman spectral representations.

The Dyson-Schwinger equations [21] [22] are an infinite system of coupled integral equations obeyed by the Green functions, first obtained for quantum electrodynamics by Dyson and Schwinger independently using different methods. Let us follow Schwinger's method. By considering the functional Fourier transform of the generating functional, one can establish the fundamental condition that the vacuum expectation value of the functional derivative of the action with respect to a field operator identically vanishes. Functionally differentiating this condition produces all the Dyson-Schwinger equations, which in turn provide exact information concerning the Green functions. However, as they always relate different amplitude Green functions, they can only be solved after being truncated to a certain level.

Gauge theories exhibit invariance under gauge group transformations and the associated S-matrix is gauge independent. This imposes certain constraints on the Green functions and requires that they obey definite

gauge transformation laws. The gauge identities, which are known as the Ward-Green-Takahashi identities [23,24,25] in abelian theories and Slavnov-Taylor [26] and Becchi-Rouet-Stora identities [27] (these two sets of identities are in fact equivalent in the sense that the one implies the other [28]) in non-abelian theories, arise as a direct consequence of the gauge invariance of the Lagrangians and equate n -point Green functions to the divergences of the $(n+1)$ -point functions. The gauge property of a Green function is described by the transformation identity, which was derived by Landau and Khalatnikov, Zumino, etc. [29,30,31] using different methods, in the context of quantum electrodynamics. The corresponding identity in nonabelian theories exists in the axial gauge [28], while that in the covariant gauge does not possess a closed form [28] [40].

Starting from the basic assumptions of quantum field theory, namely, relativistic invariance, and the existence of a Hilbert space with positive definite metric (in fact this condition is not strictly necessary), a unique vacuum, and only positive energy states, researchers have established the general structure of particle propagators as spectrally weighted integrals, known as the Källén-Lehmann-Wightman spectral representations. They were first used by Källén in 1952 [32], and the systematic derivation was given by Wightman and Lehmann in the middle of 50's in the case of quantum electrodynamics [33]. In quantum chromodynamics the widely used spectral representation for the gluon propagator is often taken in the axial gauge; however a rigorous proof of its existence is lacking so far [34].

All these exact properties of Green functions form the basis of the current version of the gauge technique [18]. By analysing Ward-

Takahashi identities and Dyson-Schwinger equations, an approximate spectral representation is obtained for the vertex with the same spectral function as that for the propagator. This vertex is then applied to truncate the Dyson-Schwinger equation for the particle propagator, leading to the Delbourgo-West integral equation for the spectral function. In principle this equation can be solved to yield a non-perturbative answer. As the vertex is designed so that it satisfies the relevant gauge identity, the gauge technique possesses the virtue of respecting that gauge identity at any stage, in contrast to other approximation schemes. The following is a brief review of the gauge technique.

1.2 HIGHLIGHTS OF THE GAUGE TECHNIQUE

Although it was introduced by Delbourgo and Salam as early as the middle 60's [17], the gauge technique lay dormant for more than a decade until in 1977 Delbourgo and West [35] revived it by abandoning the original idea of Delbourgo and Salam that the two-particle unitarity should be taken as the starting point. They discovered the simplest ansatz for the three-point photon amputated Green function and established the lowest order gauge technique equation for the spectral function in covariant gauge quantum electrodynamics. The equation was solved by them in the Landau gauge and subsequently by Slim [36] for any gauge. This work reawakened people's interests in the gauge technique and prompted extensive research in this area. Probably the most encouraging result was that the gauge technique produced the infrared behaviours of the charged particle propagators almost trivially for spinor, scalar and vector quantum electrodynamics, with the same gauge dependence in covariant gauges [37, 38]. However, in the axial gauge the method became extremely involved because of the existence of non-covariant arguments in the spectral functions [39]. In

fact the spinor axial gauge technique equation was intractable except in the infrared, and the corresponding scalar equation could only be solved for small coupling constant.

The gauge properties of the solutions of the gauge technique equations were studied by Delbourgo and Keck [40], Slim [36] and Delbourgo, Keck and Parker [41]. Because a two-point Green function obeys the Zumino identity under gauge transformations, this will impose a relationship between the spectral functions in two different gauges. Delbourgo and Keck analysed the solution of the gauge technique equation for scalar electrodynamics by transforming the Landau gauge solution into a gauge specified by a parameter 'a' using the Zumino identity, then comparing it with the solution in this gauge. It turned out that they agreed with each other precisely. Therefore gauge covariance was respected by the gauge technique in the scalar case. However, in the case of spinor electrodynamics, it was observed that the spectral function obtained by the gauge technique only maintained gauge covariance in the asymptopia. It violated the Zumino identity in intermediate energy region, but when one considered the problem perturbatively the violation disappeared.

The gauge technique was also applied to lower dimensional models. Using the naive ansatz (the one used in four dimensional electrodynamics) for the three-point vertex, Delbourgo and Shepherd [42] analysed the Schwinger model in the Feynman gauge, and found that the gauge technique could reproduce the conventional results, namely, the theory dynamically broke gauge symmetry with the vector meson acquiring a finite mass, while the spinor meson remained massless; and the spinor propagator was not affected by the interaction. However, in other gauges the naive ansatz was not consistent, as noted by Gardner [43]. To cure this inconsistency she introduced a transverse part into the ansatz. Subsequently this was

shown to be unique and complete by Delbourgo and Thompson [44]. These authors also extended the gauge technique to the $U(1)$ and $SU(N)$ Thirring models, and demonstrated that the gauge technique could be fruitfully applied to models which were not necessarily gauge theories providing they possessed gauge type identities. Applying the gauge technique to an axial model, Thompson obtained complete solutions here too [45].

Three dimensional quantum electrodynamics is plagued by infrared divergences [46]. To cancel them away, de Roo and Stam [47] modified the naive ansatz for the three-point vertex by introducing an extra transverse piece. Applying this new vertex to the Dyson-Schwinger equation they arrived at a well-behaved equation for the spectral function, and an explicit solution was obtained. Thompson and King [48] dealt with the infrared divergences in a different and simpler way: by considering the constant gauge transformations they included a constant field with the vector field, and modified the Dyson-Schwinger equations. With the naive ansatz for the three-point vertex they carried out gauge technique analysis of the theory and arrived at exactly the same results as de Roo and Stam.

Another application of the gauge technique has been to the study of dynamical symmetry breaking, where the dynamical Goldstone mechanism is combined with gauge symmetry to create masses for the gauge bosons. This can be done without introducing Higgs fields, the presence of which is the main criticism levelled at the conventional Higgs mechanism. In 1980 Delbourgo and Keck [49] investigated a chiral system comprised of a fermion coupled to vector and axial bosons with a $U(1) \times U(1)$ gauge group. The theory possessed the vector and axial sets of gauge identities. While the vector gauge identity gave the naive ansatz for the three-point vector vertex, the axial gauge identity provided an ansatz for the axial vertex incorporating a pole in the axial boson leg (indicating the presence of a

pseudo-scalar Goldstone mode). These ansätze, together with the Dyson-Schwinger equation yielded an equation for the spectral function of the spinor propagator, which reduced to the traditional spinor electrodynamics gauge technique equation in the infrared, and in the ultra-violet region reproduced the leading behaviour found by Baker and Johnson [16]. Due to its complexity the authors could not solve the equation exactly; rather they designed an approximate spectral function which behaved correctly in the infrared region and also in the ultra-violet limit providing that the axial coupling constant vanished. Analysing the axial meson self-energy with this spectral function led to a mass ratio between the fermion and axial meson in terms of the coupling constants. The mass ratio was free of any divergences, hence, improved the first Jackiw-Johnson result [50], the ultra-violet cut-off dependence of which was thought as an unsatisfactory feature. With the same strategy, Delbourgo and Kenny [51] analysed the massless $SU(2) \times U(1)$ electroweak model and established an equation for the ratio of the W^\pm mass relative to the average fermion mass of a given generation. The equation permitted two distinct solutions, and this fact was regarded as possible evidence for the existence of several fermion generations coupling to a single quartet of gauge bosons. To avoid complications such as the internal group factors in the $S(2) \times U(1)$ model, Delbourgo, Parker and Kenny [52] chose to study the simpler model, massive electrodynamics. With the aid of numerical analysis they obtained an equation for the ratio of the meson mass to that of the fermion involving a finite renormalization mass, and as in the previous case, it possessed two solutions.

For all its success, the gauge technique had an inherent ambiguity in that the original scheme of solving gauge identities to provide

approximate vertex functions did not allow the determination of the transverse components; this deficiency had been noted many times. However, in some problems transverse vertices play central roles. As we have seen, in two dimensional models the naive 'longitudinal' vertex could not produce satisfactory results in some cases, and the necessity of including transverse vertices was obvious. In four-dimensional electrodynamics, although the lowest order gauge technique was very successful, the renormalizability of the Delbourgo-West equation was not so apparent and Delbourgo [18], [39] conjectured that the transverse vertex contributions were responsible for the cancellation of the divergences. In spinor electrodynamics, it was also believed that the non-gauge covariance of the spectral function was due to the absence of the transverse vertex. To see whether these statements are true or not, one has no other choice but to include the transverse contributions into the gauge technique. Another important fact about the transverse vertex is that the anomalous magnetic moment of an electron arises from the $\sigma_{\mu\nu} k^\nu$ part of it. Therefore it is desirable to introduce transverse vertices and initiate the next stage of evolution of the gauge technique.

It was not until 1983, when King [53] for the first time modified the naive longitudinal ansatz to introduce some transversality, that a proper attack on the four-dimensional problem began. Totally relying on perturbation theory and being concerned with only the leading log effects of the electron self-energy in asymptopia, King constructed a transverse vertex, which led to a refined gauge technique equation. This equation could produce the standard infrared singularity for the fermion propagator, and in the ultra-violet region led to a solution coinciding with the renormalization-group-improved perturbation theory; however it was not

able to reproduce the perturbation results exactly to e^4 order. At much the same time, and in the context of scalar electrodynamics Parker [54] invented a new method to improve the gauge technique. Truncating the Dyson-Schwinger equations for the three-point vertex function in a manner consistent with perturbation theory, he obtained a non-perturbative transverse vertex which was correct in any momentum region. This yielded a refined spectral equation (valid only for the Feynman gauge and incorporating an arbitrary constant) which was explicitly demonstrated to be renormalizable and capable of reproducing the standard infrared result for the charged particle propagator. By properly handling the complexities of the Dirac algebras, Delbourgo and Zhang [55] succeeded in introducing non-perturbative transverse vertices to spinor electrodynamics in a different way. They obtained a new gauge technique equation which was finite, linear in the spectral function, exact to e^4 in a perturbative sense and unambiguous in contrast to Parker's equation. In infrared region it reproduced the standard solution.

Before closing this section we should mention the application of the lowest order gauge technique in chromodynamics. The success of the gauge technique in electrodynamics gave people the impression that it could also be applied to study the colour confinement problem through investigating the infrared properties of the gluon propagator. Since 1979 extensive research took place in this area [56,57,58,59,60] focusing on the pure Yang-Mills chromodynamics (with quark-gluon interactions neglected) in axial and light-cone gauges, where the ghost particles disappear and the gauge identities assume the QED-like forms [61,62]. Although different people employed different methods in analysing the problem, they mainly followed two approaches. One was to study the infrared gluon propagator immediately with the gauge technique [56, 57], the other

was to design an 'effective gluon propagator' first, then study its infrared properties [60]. But unfortunately the results obtained by different groups often contradicted one another, and West [63] even claimed that the axial gauge gluon propagator had nothing to do with colour confinement. We will come back to this problem in chapter VI. There the work done on this problem is reviewed in rather more detail, and I present my argument [64] that the gluon self-energy lies solely in the contributions of a transverse part of the vertex; therefore the lowest order gauge technique by itself and as used by all the previous authors, is not applicable to massless chromodynamics; and even Cornwall's much sounder approach can only lead to a qualitative result, that is, the generation of a dynamical gluon mass.

I.3 STRUCTURE OF THE THESIS

The original material reported in this thesis resides in Sections (III.2), (III.3), (III.4), (IV.3), (IV.4), (IV.5), (V.2), (V.3) and (VI.3).

As the preliminary chapter, chapter II examines the basic elements of the gauge technique in electrodynamics. In order to make the treatment self-contained we give a detailed derivation of the first few Dyson-Schwinger equations and the general formula for the Ward-Takahashi identities in the context of spinor electrodynamics. For this we use functional methods, details of which can be found in some excellent review articles [65, 66]. As the Dyson-Schwinger equations and Ward-Takahashi identities arise from different principles of field theories, we demonstrate explicitly the consistency of these relationships amongst Green functions. The Källén-Lehmann-Wightman spectral representations of the charged particle propagators are briefly presented out in section II.3, their derivations from the elementary assumptions of quantum field theories may be found in any one of a number of standard text

books [67, 68, 69]. The last section sketches out the treatment in axial gauges.

Chapters III, IV, V and VI form the main body of the thesis. They are concerned with the applications of the gauge technique to spinor, scalar electrodynamics and two dimensional models as well as chromodynamics, respectively. We begin chapter III with a short review of the lowest order gauge technique, then in section III.2 we develop a non-perturbative method for introducing transverse vertices in spinor electrodynamics, which is closely related to Parker's method [54] in the scalar case. Manipulating the Dyson-Schwinger equation of the three-point photon amputated Green function and its conjugate equation, we obtain an explicit expression for the Green function as a functional of other Green functions. Truncating this expression results in a spectral representation for the three-point photon amputated Green function. This spectral representation incorporates a purely transverse piece, has all the desired properties and reproduces the e^2 order perturbation theory exactly.

In section III.3 we utilize this new ansatz to truncate the Dyson-Schwinger equation for the electron propagator. This leads to a refined gauge technique equation, which is entirely free of divergences, linear in the spectral function and able to reproduce the perturbation theory up to e^4 order. Its infrared solution leads to the standard singularity of the spectral function. The new ansatz also results in a gauge invariant photon vacuum polarization. We demonstrate this explicitly up to e^4 order using perturbation theory in the final section of this chapter.

Chapter IV is devoted to scalar electrodynamics. After reviewing the lowest order gauge technique, we discuss Parker's work [54] in a little detail. By changing from the Feynman gauge (Parker's choice) to a

covariant gauge specified by the gauge parameter 'a', we arrive at a finite and unique equation. This is in contrast to Parker's equation in which a finite constant was introduced to correct the ambiguity in the finite terms caused by renormalization. In the infrared limit our equation naturally yields the well-known infrared behaviour for the spectral function. In section IV.4 we analyse the photon vacuum polarisation using perturbation theory and reach the conclusion that the transverse contributions do improve its gauge properties; it remains gauge invariant up to the order e^4 . The last section of this chapter deals with the axial gauge. We introduce an important approximation to the axial gauge technique equation which enables us to solve it in all momentum regions.

Two dimensional models are dealt with in chapter V. By reviewing the applications of gauge technique to the Schwinger model, we conclude that the gauge technique can in principle be applied to solve this model, but except for the case of Landau gauge, the solution for the spectral function can not easily be obtained in closed form; in non-covariant gauges the problem becomes even more complicated because of the existence of a non-covariant argument in the spectral function. Therefore we depart slightly from the conventional procedure by abandoning the spectral representation and solving the Dyson-Schwinger equations directly in configuration space (see Section V.2). An exact solution for the spinor particle propagator is obtained, which is valid for the Schwinger model in all linear gauges and for the Thirring model as well. In the final section we extend our solution to the finite temperature case.

Chapter VI is a self-contained chapter examining the applications of the gauge technique in chromodynamics. Section VI.1 outlines the axial gauge chromodynamics, and in section VI.2, we review the work done

in this area by other people, in particular, we present the BBZ program [57] in details. In section VI.3 we manipulate the Dyson-Schwinger equation of the gluon propagator with the aid of the Slavnov-Taylor identity and arrive at the conclusion that the radiative corrections to the gluon propagator lie solely in a transverse part of the three-gluon vertex (if the four-gluon contributions are neglected). Therefore the lowest order gauge technique is inapplicable to this problem.

We conclude the thesis with a summary and a survey of possible developments in the future. These form the final chapter.

The two appendices contain the most intricate work of this thesis. In Appendix A we evaluate the mass correction to order e^4 of a spinor-1/2 particle with mass M , employing the cut-off regularization [70] procedure, while Appendix B outlines the calculation of the mass operator of a meson with mass squared M^2 using dimensional regularization [71], [72], [73].

The references are contained at the very end of the thesis.

II. INGREDIENTS OF THE GAUGE TECHNIQUE

As a non-perturbative method for solving gauge theories, the gauge technique employs three basic tools, namely Dyson-Schwinger equations [21, 22], Ward-Takahashi identities [23, 24, 25] and the Källén-Wightman-Lehmann spectral representations [32, 33]. This chapter is devoted to the derivation of the first few Dyson-Schwinger equations and a general formula for the Ward-Takahashi identities in the context of quantum electrodynamics. We also demonstrate explicitly the consistency of the two sets of relations. At the end of the chapter we present a brief description of the Källén-Wightman-Lehmann spectral representations and sketch what happens for non-covariant gauges.

II.1 Dyson-Schwinger Equations

The temporal development of a quantum field is described by its Green functions; and the Green functions obey a system of exact equations, which, known as Dyson-Schwinger equations, were first derived by Dyson [22] and Schwinger [21] independently through different approaches. Although they certainly embody much useful information, not many interesting results have been obtained by using them and, in spite of their existence, the structure of Green functions is still mostly studied by perturbative methods. However, since the inception of gauge technique they have found extensive applications and revealed important non-perturbative features of Green functions. In the following we will derive the first few Dyson-Schwinger equations in the context of spinor electrodynamics by utilizing functional techniques [65, 66] and simply list the corresponding ones in scalar case [54]. We want to emphasize that the functional method employed here is general and can be applied to more complicated theories, such as nonabelian gauge theories, without any

essential change.

Derivation of Dyson-Schwinger equations

The Lagrangian $\mathcal{L}(x)$ of an electromagnetic field $A_\mu(x)$ interacting with an electron field $\psi(x)$ is

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \bar{\psi}(x) [i\gamma^\mu \partial_\mu - e_0 A_\mu(x) \gamma^\mu - m_0] \psi(x) \quad (1)$$

in terms of unrenormalized parameters and wave functions. As usual, $F_{\mu\nu}(x)$ denotes the field strength tensor and is defined by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \quad (2)$$

Introducing a spinor source $J(x)$ for the electron field $\psi(x)$ and a vector source $J_\mu(x)$ for the photon field $A_\mu(x)$, we define the action

$$S(x) = \int d^4x [\mathcal{L}(x) - \bar{J}(x)\psi(x) - \bar{\psi}(x)J(x) - J_\mu(x) A^\mu(x)] \quad (3)$$

in which renormalization may be carried out by including counterterms. These counterterms will not, however, be spelled out explicitly until the end of the derivation for the sake of simplicity. Now the generating functional of the full Green functions may be expressed as

$$Z[\bar{J}, J, J_\mu] = \int [d\psi d\bar{\psi} dA_\mu] \exp[iS] \prod_x \delta(F(x) - \lambda(x)) \cdot \det\left(\frac{\delta F}{\delta \Lambda}\right) \quad (4)$$

following the Fadeev-Popov procedure [74]. Here we have adopted the standard notation: $\Lambda(x)$ is the function parameterizing gauge transformations and the condition

$$F(x) - \lambda(x) = 0$$

specifies a particular gauge.

Due to the λ independence of $Z[\bar{J}, J, J_\mu]$ we can introduce the function

$$\exp\left[-\frac{i}{2a_0} \int d^4x \lambda^2(x)\right]$$

into the right-hand-side of (4) and functionally integrate it over λ .

This results in

$$Z[\bar{J}, J, J_\mu] = \int [d\psi d\bar{\psi} dA_\mu] \exp \left\{ i \left[S - \frac{1}{2a_0} \int d^4x F^2(x) \right] \right\} \det\left(\frac{\delta F}{\delta \Lambda}\right), \quad (5)$$

with a_0 known as the unrenormalized gauge parameter. Under the gauge transformation of $A_\mu(x)$ given by

$$\delta A_\mu(x) = \partial_\mu \Lambda(x) \quad (6)$$

the particular function

$$F(x) = \partial_\mu A^\mu(x) \quad (7)$$

which specified a covariant gauge, leads to

$$\det\left(\frac{\delta F}{\delta \Lambda}\right) = \det(\partial^\mu \partial_\mu) \quad (8)$$

Equation (8) is obviously independent of both the integration variables and the sources, and hence the determinant (8) can be removed from the integral and absorbed into the normalization constant. Thus (5) simply assumes the form (up to an unimportant normalization constant),

$$Z[\bar{J}, J, J_\mu] = \int [d\psi d\bar{\psi} dA_\mu] \exp(iE) \quad (9)$$

where E is a new action given by

$$E = S - \frac{1}{2a_0} \int d^4x (\partial_\mu A^\mu(x))^2 \quad (10)$$

The second integral on the right-hand-side is generally known as the gauge fixing term. In our case we obtain the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0$$

in the $a_0 \rightarrow 0$ limit.

The momentum space version of (9), with which we will work exclusively later on, reads

$$Z[\bar{J}, \tilde{J}, \tilde{J}_\mu] = \int [d\tilde{\psi} d\tilde{\bar{\psi}} d\tilde{A}_\mu] \exp(iE) \quad (9')$$

with E given by ($\bar{d}^4 k = d^4 k / (2\pi)^4$ here and throughout)

$$E = \int \bar{d}^4 k \left\{ -\frac{1}{4} \tilde{F}_{\mu\nu}(k) \tilde{F}^{\mu\nu}(-k) + \tilde{\bar{\psi}}(k)(\not{k} - m_0)\tilde{\psi}(k) - e_0 \int \bar{d}^4 k' \tilde{\bar{\psi}}(k) \tilde{A}(k-k') \tilde{\psi}(k') \right. \\ \left. - \tilde{J}(k)\tilde{\psi}(k) - \tilde{\bar{\psi}}(k)\tilde{J}(k) - \tilde{J}_\mu(k)\tilde{A}^\mu(-k) - \frac{1}{2a_0} [k_\mu \tilde{A}^\mu(k) k_\nu \tilde{A}^\nu(-k)] \right\} \quad (11)$$

and $\tilde{A}_\mu(k)$, $\tilde{J}_\mu(k)$ etc. denoting the Fourier transforms of $A_\mu(x)$, $J_\mu(x)$ respectively. In the following we will omit the tildes.

Both the connected Green function generating functional $W[\bar{J}, J, J_\mu]$ and the one-particle irreducible vertex generating functional $\Gamma[\psi, \bar{\psi}, A_\mu]$ originate in $Z[\bar{J}, J, J_\mu]$. We will study Γ in the next section; here we merely focus on W , which is defined by

$$W[\bar{J}, J, J_\mu] = -i \ln Z[\bar{J}, J, J_\mu] \quad (12)$$

Needless to say, terms independent of the external sources might be introduced into W through different normalization schemes for Z , but they are irrelevant to the Green functions, which are the functional derivatives of W with respect to the sources.

Differentiating $W[\bar{J}, J, J_\mu]$ twice with respect to the vector source and then setting all the sources to zero leads to the photon propagator D (again $\bar{\delta}(k+k') = (2\pi)^4 \delta^4(k+k')$ here and throughout):

$$\text{i.e.} \quad \frac{\delta^2 W[0,0,0]}{\delta J_\mu(-k) \delta J_\nu(-k')} = -D^{\mu\nu}(k) \bar{\delta}(k+k') \quad (13)$$

The electron propagator can be obtained similarly

$$\frac{\delta^2 W[0,0,0]}{\delta \bar{J}(p') \delta J(p)} = -S(p) \bar{\delta}(p'-p) \quad (14)$$

The general formula

$$\begin{aligned}
& F^{\mu_1 \dots \mu_n}(p', p; k_1, \dots, k_n) \bar{\delta}(p' - p - k_1 - \dots - k_n) \\
& = i^{n+3} \frac{\delta^{(n+2)} W[0, 0, 0]}{\delta \bar{J}(p') \delta J(p) \delta J_{\mu_1}(k_1) \dots \delta J_{\mu_n}(k_n)} \quad (15)
\end{aligned}$$

gives the $(n+2)$ -point connected Green functions with n photon lines. The corresponding photon amputated amplitude $G_{\nu_1 \dots \nu_n}$ is defined by

$$\begin{aligned}
& F^{\mu_1 \dots \mu_n}(p', p; k_1, \dots, k_n) \\
& = i^{n+3} (-e_0)^n \cdot D^{\mu_1 \nu_1}(k_1) \dots D^{\mu_n \nu_n}(k_n) G_{\nu_1 \dots \nu_n}(p', p; k_1, \dots, k_n). \quad (16)
\end{aligned}$$

In both (15) and (16) the momenta before the semi-colon correspond to charged particles, and the ones after it to the photons. Except for the very first momentum, p' , which is "outgoing", all the momenta are "incoming". They are linearly dependent as a result of momentum conservation, a fact which is expressed by the attached δ -function.

Now we are ready to derive the Dyson-Schwinger equations. Our starting point is that the vacuum expectation value of a functional derivative of E with respect to any one of the fields vanishes, for example,

$$\int [d\psi d\bar{\psi} dA_\mu] \left[\frac{\delta}{\delta \psi(p)} \exp(iE) \right] = 0, \quad (17)$$

which was first discovered by Schwinger [21]. We should nevertheless pay particular attention to the order of the quantities. For instance, in (17) the expression in the square brackets should be understood as the functional derivative of E with respect to $\psi(p)$ from the right. If due care with respect to order is not taken inconsistencies will result.

Writing out the explicit expression for (17), we have

$$\int [d\psi d\bar{\psi} dA_\mu] \{ \bar{\psi}(p)(\not{p} - m_0) - e_0 \int d^4k \bar{\psi}(k) A(k-p) - \bar{J}(p) \} \exp(iE) = 0. \quad (18)$$

Then applying the following identities

$$i \frac{\delta Z}{\delta J(p)} = \int [d\psi d\bar{\psi} dA_\mu] \bar{\psi}(p) \exp(iE) \quad (19)$$

$$i \frac{\delta Z}{\delta J_\mu(-k)} = \int [d\psi d\bar{\psi} dA_\mu] A^\mu(k) \exp(iE) \quad (20)$$

to (18), we arrive at

$$\{ i \frac{\delta}{\delta J(p)} (\not{p} - m_0) + e_0 \int d^4k \frac{\delta^2}{\delta J^\mu(k) \delta J(p-k)} \gamma^\mu - \bar{J}(p) \} Z[\bar{J}, J, J_\mu] = 0 \quad (21)$$

which can be rewritten, in terms of W, as

$$\begin{aligned} & - \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J(p)} (\not{p} - m_0) + e_0 \int d^4k \left\{ i \frac{\delta^2 W[\bar{J}, J, J_\mu]}{\delta J^\mu(k) \delta J(p-k)} \right. \\ & \quad \left. - \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J^\mu(k)} \cdot \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J(p-k)} \right\} \gamma^\mu \\ & - \bar{J}(p) = 0. \end{aligned} \quad (22)$$

(22) is one of our basic equations for generating all the Dyson-Schwinger equations. After once differentiation with respect to $\bar{J}(p')$ from the left, it becomes

$$\begin{aligned} & - \frac{\delta^2 W[\bar{J}, J, J_\mu]}{\delta \bar{J}(p') \delta J(p)} (\not{p} - m_0) + e_0 \int d^4k \left\{ i \frac{\delta^3 W[\bar{J}, J, J_\mu]}{\delta J^\mu(k) \delta \bar{J}(p') \delta J(p-k)} \right. \\ & \quad \left. - \frac{\delta^2 W[\bar{J}, J, J_\mu]}{\delta J^\mu(k) \delta \bar{J}(p')} \cdot \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J(p-k)} - \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J^\mu(k)} \cdot \frac{\delta^2 W[\bar{J}, J, J_\mu]}{\delta \bar{J}(p') \delta J(p-k)} \right\} \gamma^\mu - \bar{\delta}(p' - p) = 0. \end{aligned} \quad (23)$$

When we set J and \bar{J} to zero, charge conservation requires that the second term in the brackets vanish identically, so (23) reduces to

$$\begin{aligned} & - \frac{\delta^2 W[0, 0, J_\mu]}{\delta \bar{J}(p') \delta J(p)} (\not{p} - m_0) + e_0 \int d^4k \left\{ i \frac{\delta^3 W[0, 0, J_\mu]}{\delta J^\mu(k) \delta \bar{J}(p') \delta J(p-k)} \right. \\ & \quad \left. - \frac{\delta W[0, 0, J_\mu]}{\delta J^\mu(k)} \frac{\delta^2 W[0, 0, J_\mu]}{\delta \bar{J}(p') \delta J(p-k)} \right\} \gamma^\mu - \bar{\delta}(p' - p) = 0. \end{aligned} \quad (24)$$

In vanishing J_μ limit, the equation

$$\bar{\delta}(p'-p)S(p)(\not{p}-m_0)-ie_0^2 \int \bar{d}^4k G_\nu(p',p-k;k)\gamma_\mu D^{\mu\nu}(k)\bar{\delta}(p'-p)-\bar{\delta}(p'-p)=0 \quad (25a)$$

is deduced from (24) by recalling the definitions (13) - (16) and

$$\frac{\delta^n W[0,0,0]}{\delta J_{\mu_1}(k_1) \dots \delta J_{\mu_n}(k_n)} \equiv 0 \quad (26)$$

n is a positive odd integer, and equation (26) comes from the charge conjugation invariance of our theory. Integrating (25a) over p' result in the familiar equation

$$S(p)(\not{p}-m_0) - ie_0^2 \int \bar{d}^4k G_\mu(p,p-k;k)\gamma_\nu D^{\mu\nu}(k) = 1 \quad (25b)$$

which is the first one of the infinite system of Dyson-Schwinger equations and relates the charged particle propagator to the three-point Green function. To obtain the other equations we differentiate (24) again with respect to $J_\mu(q)$. One more differentiation gives

$$\begin{aligned} & - \frac{\delta^3 W[0,0,J_\mu]}{\delta J^\nu(q)\delta \bar{J}(p')\delta J(p)} (\not{p}-m_0) + e_0 \int \bar{d}^4k \left\{ i \frac{\delta^4 W[0,0,J_\mu]}{\delta J^\nu(q)\delta J^\mu(k)\delta \bar{J}(p')\delta J(p-k)} \right. \\ & - \frac{\delta^2 W[0,0,J_\mu]}{\delta J^\nu(q)\delta J^\mu(k)} \cdot \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p')\delta J(p-k)} - \frac{\delta W[0,0,J_\mu]}{\delta J^\mu(k)} \\ & \left. \times \frac{\delta^3 W[0,0,J_\mu]}{\delta J^\nu(q)\delta \bar{J}(p')\delta J(p-k)} \right\} \gamma^\mu = 0 \quad (27) \end{aligned}$$

Setting the current J_μ to zero in (27) we arrive at

$$\begin{aligned} & D_\nu^{\nu'}(q)G_\nu(p',p;q)(\not{p}-m_0)\bar{\delta}(p'-p-q)+ie_0^2 \int \bar{d}^4k D_\nu^{\nu'}(q)D_\mu^{\mu'}(k)G_{\nu'\mu}(p',p-k;q,k)\gamma^\mu \\ & \times \bar{\delta}(p+q-p') - \int \bar{d}^4k D_{\mu\nu}(k)\bar{\delta}(q+k)S(p')\bar{\delta}(p-k-p')\gamma^\mu = 0 \quad (28a) \end{aligned}$$

which can be cast into the neat form

$$G_\nu(p',p;p'-p)(\not{p}-m_0)+ie_0^2 \int \bar{d}^4k G_{\nu\sigma}(p',p-k;p'-p,k)D^{\sigma\mu}(k)\gamma_\mu - S(p')\gamma_\nu = 0 \quad (28b)$$

merely by integrating the δ -functions away and getting rid of the overall

$D(q)$ factor.

Further differentiation of (27) with respect to $J_\mu(1)$ makes it more complicated

$$\begin{aligned}
 & \frac{\delta^4 W[0,0,J_\mu]}{\delta J^\mu(\ell) \delta J^\nu(q) \delta \bar{J}(p') \delta J(p)} (\not{p}-m_0) + e_0 \int \bar{d}^4 k \left\{ i \frac{\delta^5 W[0,0,J_\mu]}{\delta J^\mu(\ell) \delta J^\nu(q) \delta \bar{J}(p') \delta J(p-k) \delta J^\rho(k)} \right. \\
 & - \frac{\delta^3 W[0,0,J_\mu]}{\delta J^\mu(\ell) \delta J^\nu(q) \delta J^\rho(k)} \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p-k)} - \frac{\delta^2 W[0,0,J_\mu]}{\delta J^\nu(q) \delta J^\rho(k)} \cdot \frac{\delta^3 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p-k) \delta J^\mu(\ell)} \\
 & - \frac{\delta^2 W[0,0,J_\mu]}{\delta J^\mu(\ell) \delta J^\rho(k)} \cdot \frac{\delta^3 W[0,0,J_\mu]}{\delta J^\nu(q) \delta \bar{J}(p') \delta J(p-k)} \\
 & \left. - \frac{\delta W[0,0,J_\mu]}{\delta J^\rho(k)} \frac{\delta^4 W[0,0,J_\mu]}{\delta J^\mu(\ell) \delta J^\nu(q) \delta \bar{J}(p') \delta J(p-k)} \right\} \gamma^\rho = 0 .
 \end{aligned} \tag{29}$$

Going through the same procedure as we did in deriving (25b) and (28b) we can obtain

$$\begin{aligned}
 & G_{\mu\nu}(p', p; \ell, p'-p-\ell) (\not{p}-m_0) + i e_0^2 \int \bar{d}^4 k G_{\mu\nu\rho}(p', p-k; \ell, p'-p-\ell, k) D^{\rho\sigma}(k) \gamma_\sigma \\
 & + G_\mu(p', p'-\ell; \ell) \gamma_\nu + G_\nu(p', p+\ell; p'-p-\ell) \gamma_\mu = 0 .
 \end{aligned} \tag{30}$$

It is possible in principle to work out the higher order derivatives of (29) with respect to J_μ 's, and obtain other equations. However, it becomes so involved as to be unmanageable a few orders higher.

The three equations we obtained so far indicate a general feature of the Dyson-Schwinger equations. They always equate n -point Green functions to Green functions of more than n -points. Therefore the set of equations is an open system, and in order to obtain any useful information from them truncation is needed.

Apart from (17) we can also consider the equation

$$\begin{aligned}
& \int [d\psi d\bar{\psi} dA_\mu] \left[\frac{\delta}{\delta \bar{\psi}(p')} \exp(iE) \right] \\
& = i \int [d\psi d\bar{\psi} dA_\mu] \{ (\not{p}' - m_0) \psi(p') - e_0 \int \bar{d}^4 k \gamma^\mu A_\mu(p'-k) \psi(k) - J(p') \} \exp(iE) = 0,
\end{aligned} \tag{31}$$

which is Schwinger's condition. Some simple manipulations lead us to

$$\begin{aligned}
& - (\not{p}' - m_0) \frac{\delta W[\bar{J}, J, J_\mu]}{\delta \bar{J}(p')} + e_0 \int \bar{d}^4 k \gamma^\rho \left\{ i \frac{\delta^2 W[\bar{J}, J, J_\mu]}{\delta J^\rho(k) \delta \bar{J}(p'+k)} \right. \\
& \quad \left. - \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J^\rho(k)} \cdot \frac{\delta W[\bar{J}, J, J_\mu]}{\delta \bar{J}(p'+k)} \right\} - J(p') = 0, \tag{32}
\end{aligned}$$

the charge conjugate of equation (22). This will generate another system of Dyson-Schwinger equations which are of the same structure as the ones we obtained above but conjugated. To start we differentiate (32) with respect to $J(p)$ from the right, and then set the spinor sources to zero, this results in

$$\begin{aligned}
& - (\not{p}' - m_0) \frac{\delta^2 W[0, 0, J_\mu]}{\delta \bar{J}(p') \delta J(p)} + e_0 \int \bar{d}^4 k \gamma^\rho \left\{ i \frac{\delta^2 W[0, 0, J_\mu]}{\delta J^\rho(k) \delta \bar{J}(p'+k) \delta J(p)} \right. \\
& \quad \left. - \frac{\delta W[0, 0, J_\mu]}{\delta J^\rho(k)} \cdot \frac{\delta^2 W[0, 0, J_\mu]}{\delta \bar{J}(p'+k) \delta J(p)} \right\} - \bar{\delta}(p'-p) = 0, \tag{33}
\end{aligned}$$

which produces an equation conjugate to (25b), in the $J_\mu = 0$ limit,

$$(\not{p} - m_0) S(p) - ie_0^2 \int \bar{d}^4 k \gamma_\nu G_\mu(p+k, p; k) D^{\mu\nu}(k) = 1. \tag{34}$$

We can manipulate (33) in exactly the same way as we dealt with (23) to obtain the equations conjugate to (28b) and (30). They read

$$\begin{aligned}
& (\not{p}' - m_0) G_\mu(p', p; p'-p) + ie_0^2 \int \bar{d}^4 k \gamma_\nu G_{\mu\sigma}(p'+k, p; p'-p, k) D^{\nu\sigma}(k) - \gamma_\mu S(p) = 0, \tag{35} \\
& (\not{p}' - m_0) G_{\nu\mu}(p', p; \ell, p'-p-\ell) + ie_0^2 \int \bar{d}^4 k \gamma_\rho G_{\nu\mu\rho}(p'+k, p; \ell, p'-p-\ell, k) D^{\rho\rho'}(k) \\
& \quad + \gamma_\nu G_\mu(p'-\ell, p; p'-p-\ell) + \gamma_\mu G_\nu(p+\ell, p; \ell) = 0. \tag{36}
\end{aligned}$$

We have succeeded now in deriving the first three Dyson-Schwinger equations and their corresponding conjugate equations. One point concerning the derivations should be cleared up: although the gauge fixing term in (10), specifying the covariant gauge, was included at the very beginning, we did not refer to it at all. Hence the forms of the equations are independent of the gauge choice; they are gauge covariant.

The photon propagator Dyson-Schwinger equation comes from the equation

$$\int [d\psi d\bar{\psi} dA_\mu] \left[\frac{\delta}{\delta A_\mu(k)} \exp(iE) \right] = 0 . \quad (37)$$

We will omit the details of its derivation because nothing new is involved, and merely state the result

$$D_{\mu\nu}^{-1}(k) = D_{\mu\nu}^{(0)-1}(k) + ie_0^2 \int d^4p \operatorname{tr}[G_\mu(p, p+k; -k) \gamma_\nu] . \quad (38)$$

Here

$$D_{\mu\nu}^{(0)}(k) = \frac{1}{k^2 + i0^+} [-g_{\mu\nu} + (1-a_0) \frac{k_\mu k_\nu}{k^2 + i0^+}] , \quad (39)$$

is the unrenormalized bare photon propagator in a covariant gauge.

Renormalization

In the above equations, all the quantities are unrenormalized - we did not introduce counterterms into our theory in order to avoid cumbersome notations. Because renormalization is not our main concern here, we will not analyse the problems that introduces in great details. Rather we will content ourselves with sketching the modifications to the equations necessitated by multiplicative renormalizations of Green functions. For details of renormalization theory, we refer to the text books [11] [67, 68] and review articles [71] [75].

First of all we consider the renormalization constant $Z_{G(n)}$ which renormalizes the n -point photon amputated Green function according to

$$G^{(n)} = Z_{G^{(n)}} \cdot G_R^{(n)} \quad (40)$$

In this equation the R subscript is used to denote the renormalized quantities. Recalling that

$$A_\mu = Z_A^{-1/2} A_{R\mu}, \quad \psi = Z_\psi^{-1/2} \psi_R, \quad e_R^2 = Z_A e_0^2 \quad (41)$$

and the definitions (15) and (16) we obtain the simple relationship

$$Z_{G^{(n)}} = Z_\psi \quad (42)$$

Now we are ready to renormalize the Dyson-Schwinger equations. Replacing all the quantities with their renormalized values in (25b), (34) and (38) results in

$$S_R(p)(\not{p}-m_0)-ie_R^2 \int \bar{d}^4 k G_{R\mu}(p,p-k;k) \gamma_\nu D_R^{\mu\nu}(k) = Z_\psi^{-1} \quad (43a)$$

$$(\not{p}-m_0)S_R(p)-ie_R^2 \int \bar{d}^4 k \gamma_\mu G_{R\nu}(p+k,p;k) D_R^{\mu\nu}(k) = Z_\psi^{-1} \quad (43b)$$

$$D_{R\mu\nu}^{-1}(k) = Z_A D_{\mu\nu}^{(0)-1}(k) + ie_R^2 Z_\psi \int \bar{d}^4 p \text{tr}[G_{R\mu}(p,p+k;-k) \gamma_\nu] \quad (44)$$

The other equations are not modified, but we list them anyway for completeness.

$$(\not{p}'-m_0)G_{R\mu}(p',p;p'-p)+ie_R^2 \int \bar{d}^4 k \gamma_\sigma G_{R\mu\rho}(p'+k,p;p'-p,k) D_R^{\sigma\rho}(k) - \gamma_\mu S_R(p) = 0 \quad (45a)$$

$$G_{R\mu}(p',p;p'-p)(\not{p}-m_0)+ie_R^2 \int \bar{d}^4 k G_{R\mu\rho}(p',p-k;p'-p,k) \gamma_\sigma D_R^{\sigma\rho}(k) - S_R(p') \gamma_\mu = 0 \quad (45b)$$

$$\begin{aligned} &(\not{p}'-m_0)G_{R\mu\nu}(p',p;\ell,p'-p-\ell)+ie_R^2 \int \bar{d}^4 k \gamma_\sigma G_{R\mu\nu\rho}(p'+k,p;\ell,p'-p-\ell,k) D_R^{\sigma\rho}(k) \\ &+ \gamma_\mu G_{R\nu}(p'-\ell,p;p'-p-\ell) + \gamma_\nu G_{R\mu}(p+\ell,p;\ell) = 0 \end{aligned} \quad (46a)$$

$$\begin{aligned} &G_{R\mu\nu}(p',p;\ell,p'-p-\ell)(\not{p}-m_0)+ie_R^2 \int \bar{d}^4 k G_{R\mu\nu\rho}(p',p-k;\ell,p'-p-\ell,k) \gamma_\sigma D_R^{\sigma\rho}(k) \\ &+ G_{R\mu}(p',p'-\ell;\ell) \gamma_\nu + G_{R\nu}(p',p+\ell;p'-p-\ell) \gamma_\mu = 0 \end{aligned} \quad (46b)$$

As one can see, there remain infinities in the above equations, but they will cancel among themselves to assure the finiteness of our theory.

Scalar Electrodynamics

Now we turn to consider another theoretical model, scalar electrodynamics, governed by the Lagrangian

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \{[i\partial_\mu - e_0 A_\mu(x)]\phi(x)\}^\dagger \{[i\partial^\mu - e_0 A^\mu(x)]\phi(x)\} - m_0^2 \phi^\dagger(x) \phi(x) ,$$

with all the parameters and wave functions unrenormalized. Parallel calculations lead us to the following Dyson-Schwinger equations

$$\begin{aligned} (p^2 - m_0^2) \Delta_R(p) - ie_R^2 \int \bar{d}^4 k (2p+k)_\mu G_{R\nu}(p+k, p; k) D_R^{\mu\nu}(k) \\ + e_R^4 \int \bar{d}^4 k \bar{d}^4 k' g_{\mu\nu} D_R^{\mu\alpha}(k') D_R^{\nu\beta}(k) G_{R\alpha\beta}(p+k+k', p; k', k) + \text{photon tadpole} \\ = Z_\phi^{-1} \end{aligned} \quad (47a)$$

$$\begin{aligned} (p^2 - m_0^2) \Delta_R(p^2) - ie_R^2 \int \bar{d}^4 k (2p-k)_\mu G_{R\nu}(p, p-k, k) D_R^{\mu\nu}(k) \\ + e_R^4 \int \bar{d}^4 k \bar{d}^4 k' g_{\mu\nu} D_R^{\mu\alpha}(k') D_R^{\nu\beta}(k) G_{R\alpha\beta}(p, p-k-k'; k', k) + \text{photon tadpole} \\ = Z_\phi^{-1} \end{aligned} \quad (47b)$$

$$\begin{aligned} [(p'^2 - m_0^2) + \text{photon tadpole term}] G_{R\rho}(p', p) = (p' + p)_\rho \Delta_R(p) \\ - 2ie_R^2 \int \bar{d}^4 t g_{\rho\sigma} D_R^{\sigma\pi}(t) G_{R\pi}(p+t, p) - ie_R^2 \int \bar{d}^4 t (2p'+t)_\mu D_R^{\mu\sigma}(t) G_{R\sigma\rho}(p'+t, p; t, p'-p) \\ + e_R^4 \int \bar{d}^4 t \bar{d}^4 t' g_{\mu\sigma} D_R^{\mu\lambda}(t') D_R^{\sigma\pi}(t) G_{R\lambda\pi\rho}(p'+t+t', p; t', t, p'-p) \end{aligned} \quad (48a)$$

$$\begin{aligned} [(p'^2 - m_0^2) + \text{photon tadpole term}] G_{R\rho}(p', p) = (p' + p)_\rho \Delta_R(p') \\ - 2ie_R^2 \int \bar{d}^4 t g_{\rho\sigma} D_R^{\sigma\pi}(t) G_{R\pi}(p', p'-t) - ie_R^2 \int \bar{d}^4 t (2p-t)_\mu D_R^{\mu\sigma}(t) G_{R\sigma\rho}(p', p-t; t, p'-p) \\ + e_R^4 \int \bar{d}^4 t \bar{d}^4 t' g_{\mu\sigma} D_R^{\mu\lambda}(t') D_R^{\sigma\pi}(t) G_{R\lambda\pi\rho}(p', p-t-t'; t', t, p'-p) \end{aligned} \quad (48b)$$

$$\begin{aligned} D_{R\mu\nu}^{-1}(t) = Z_A D_{\mu\nu}^{(0)-1}(t) - iZ_\phi e^2 \int \bar{d}^4 p (2p+t)_\mu G_{R\nu}(p, p+t) \\ + 2iZ_\phi e^2 \int \bar{d}^4 p g_{\mu\nu} \Delta(p^2) + 2Z_\phi e^4 \int \bar{d}^4 p \bar{d}^4 p' D_{R\mu\sigma}(p') G_{\sigma\nu}(p, p+t+p'; -t, -p') g^{\sigma\rho} \end{aligned} \quad (49)$$

where $\Delta_R(p)$ is the meson propagator and G_R 's are photon amputated Green functions. The parameters e_R and m_0 are respectively the meson's charge and its bare mass. As before we use "R" to indicate renormalized quantities. In latter chapters the R will be omitted whenever this does not cause confusion.

II.2 WARD-GREEN-TAKAHASHI IDENTITIES

Derivation of the identities

The Lagrangian density given in (1) exhibits an invariance under the local $U(1)$ group

$$U(x) = e^{-ie_0\Lambda(x)} \quad (50)$$

with the fields transforming according to

$$\begin{aligned} A_\mu(x) &\rightarrow A_\mu + \partial_\mu \Lambda(x) \\ \psi(x) &\rightarrow e^{-ie_0\Lambda(x)} \psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x) e^{+ie_0\Lambda(x)} \end{aligned} \quad (51)$$

This property, known as gauge invariance, is a general feature of the gauge theories. Though various theories possess different gauge groups, some theories feature complicated nonabelian groups such as the $SU(3)$ color group of quantum chromodynamics and the spontaneously broken $SU(2) \times U(1)$ group for the standard electroweak model, etc. This invariance allows us to derive certain gauge identities, the Ward-Green-Takahashi identities [23,24,25] for abelian field theories and Slavnov-Taylor [26] and BRS [27] identities in nonabelian case, connecting various Green functions, which in turn make for relationships among different renormalization constants and, as a consequence, permit multiplicative renormalizations. In this section we will derive a general formula for the gauge identities of the $U(1)$ gauge theory, electrodynamics.

We write out the infinitesimal version of (51)

$$\begin{aligned}
 \delta A_\mu(x) &= \partial_\mu \Lambda(x) \\
 \delta \psi(x) &= -ie_0 \Lambda(x) \psi(x) \\
 \delta \bar{\psi}(x) &= ie_0 \Lambda(x) \bar{\psi}(x) .
 \end{aligned} \tag{52}$$

Note that the infinitesimal scalar function $\Lambda(x)$ is real. In momentum space, (52) assumes the form

$$\begin{aligned}
 \delta A_\mu(k) &= -ik_\mu \Lambda(k) \\
 \delta \psi(p) &= -ie_0 \int d^4k \Lambda(k) \psi(p-k) \\
 \delta \bar{\psi}(p) &= ie_0 \int d^4k \Lambda(k) \bar{\psi}(p+k) .
 \end{aligned} \tag{53}$$

Evidently the last identity is the conjugate of the second one, rather than the Fourier transform of the third identity in (52).

The gauge function $\Lambda(k)$ is supposed to be functionally independent of all the fields in our theory, therefore it is trivial to show that the integration measure of (9') is invariant under the gauge transformation, that is

$$[d\psi^\Lambda d\bar{\psi}^\Lambda dA_\mu^\Lambda] = [d\psi d\bar{\psi} dA_\mu] , \tag{54}$$

up to $O(\Lambda^2)$ terms. Thus the only gauge variant pieces of $Z[\bar{J}, J, J_\mu]$, as given by (9'), are the gauge fixing term and the source terms in the action E , which, under the gauge transformation, transforms like

$$\begin{aligned}
 E^\Lambda &= E + \delta E \\
 &= E - \int d^4k \{ -ie_0 \bar{J}(k) \int d^4t \Lambda(t) \psi(k-t) + ie_0 \int d^4t \Lambda(t) \bar{\psi}(k+t) J(k) \\
 &\quad - ik_\mu J^\mu(-k) \Lambda(k) - \frac{i}{a_0} k^2 k^\mu A_\mu(-k) \Lambda(k) \} .
 \end{aligned} \tag{55}$$

Gauge invariance of the vacuum functional $Z[\bar{J}, J, J_\mu]$ gives rise to the fundamental equation

$$0 = \frac{\delta Z[\bar{J}, J, J_\mu]}{\delta \Lambda(k)} = -ie_0 \int d^4t \left\{ \bar{J}(t) \frac{\delta Z[\bar{J}, J, J_\mu]}{\delta \bar{J}(t-k)} - \frac{\delta Z[\bar{J}, J, J_\mu]}{\delta J(t+h)} J(t) \right\} \\ - \left\{ k^\mu J_\mu(-k) + i \frac{k^2}{a_0} k_\mu \frac{\delta}{\delta J_\mu(k)} \right\} Z[\bar{J}, J, J_\mu] \quad (56)$$

which can be re-expressed in terms of the connected Green function generating functional $W[\bar{J}, J, J_\mu]$ as

$$e_0 \int d^4t \left\{ \bar{J}(t) \frac{\delta W[\bar{J}, J, J_\mu]}{\delta \bar{J}(t-k)} - \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J(t+k)} J(t) \right\} + \frac{k^2}{a_0} k_\mu \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J_\mu(k)} \\ - k^\mu J_\mu(-k) = 0 \quad (57)$$

There are two ways which we can follow in order to study (57). One way is to analyse it by differentiating successively with respect to the external sources; this produces a series of identities equating various connected Green functions. The other way is to utilize Legendre transforms to convert (57) into an equation in terms of the one-particle-irreducible vertex generating functional. This equation, after being differentiated with respect to vacuum expectation values of the field operators, produces gauge identities relating different vertex functions. As our latter work is exclusively concerned with photon amputated Green functions, we will use the first approach, but a brief sketch of the second method is given at the very end of this section.

Differentiating (57) with respect to the external current $J_\mu(k')$, we arrive, in the vanishing \bar{J} and J limit, at the equation

$$\frac{k^2}{a_0} k_\mu \frac{\delta^2 W[0, 0, J_\mu]}{\delta J_\mu(k) \delta J_\nu(k')} - k^\mu \bar{\delta}(k'+k) = 0 \quad (58)$$

The photon propagator gauge identity follows from (58) immediately by recalling (13),

$$\frac{k^2}{a_0} k^\mu D_{\mu\nu}(k) + k_\nu = 0 \quad (59)$$

(59) reveals an extremely important fact about the photon propagator, namely, that its longitudinal part is not affected by radiative corrections. We will return to this fact later when we consider the vacuum polarization.

Now we differentiate (57) twice with respect to $\bar{J}(p')$ from the left and $J(p)$ from the right, then set the external sources \bar{J} and J to zero. All the other terms vanish in the resultant equation leaving us with

$$e_0 \left\{ \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p'-k) \delta J(p)} - \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p+k)} \right\} + \frac{k^2}{a_0} k^\mu \frac{\delta^3 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p) \delta J_\mu(k)} = 0. \quad (60)$$

Recalling the definitions (13)-(16), we can obtain

$$e_0 [S(p') - S(p)] \bar{\delta}(p' - k - p) + (-e_0) \cdot \frac{k^2}{a_0} k_\mu D^{\mu\nu}(k) G_\nu(p', p; k) \bar{\delta}(p' - k - p) = 0 \quad (61)$$

from (60). Then when we simplify (61) with the aid of (59) and eliminate the overall e_0 factor and the δ -function, an elegant identity is established

$$k^\nu G_\nu(p', p; k) = S(p) - S(p'), \quad (62)$$

with $k = p' - p$. Recalling that the G_μ function is nothing but the three-point one-particle irreducible vertex with two electron lines, that is

$$G_\mu(p', p; k) = S(p') \Gamma_\mu(p', p; k) S(p), \quad (63)$$

we can rewrite (62) as

$$k^\mu \Gamma_\mu(p', p; k) = S^{-1}(p') - S^{-1}(p). \quad (64)$$

Equation (64) is the identity first obtained by Green [24] and Takahashi [25] using perturbative expansion and canonical methods in the 1950's. It is nowadays known to be the first of a series of gauge identities. Its vanishing photon momentum limit gives us the original Ward identity

[23]

$$\Gamma_{\mu}(p,p;0) = \frac{\partial S^{-1}(p)}{\partial p^{\mu}} . \quad (65)$$

The equality of the vertex renormalization constant Z_T and the inverse electron propagator renormalization constant follows from (65) naturally, a fact which greatly simplifies the renormalization of quantum electrodynamics.

To explore (60) further we differentiate it with respect to $J_{\nu}(\ell)$

$$\begin{aligned} e_0 \left\{ \frac{\delta^3 W[0,0,J_{\mu}]}{\delta \bar{J}(p'-k) \delta J(p) \delta J_{\nu}(\ell)} - \frac{\delta^3 W[0,0,J_{\mu}]}{\delta \bar{J}(p') \delta J(p+k) \delta J_{\nu}(\ell)} \right\} \\ + \frac{k^2}{a_0} k_{\mu} \frac{\delta^4 W[0,0,J_{\mu}]}{\delta \bar{J}(p') \delta J(p) \delta J_{\mu}(k) \delta J_{\nu}(\ell)} = 0 , \quad (66) \end{aligned}$$

then set J_{μ} to zero. The resultant equation reads

$$\begin{aligned} e_0(-e_0) \cdot D^{\nu\nu'}(\ell) [G_{\nu}(p'-k,p;\ell) - G_{\nu}(p',p+k;\ell)] \bar{\delta}(p'-k-p-\ell) \\ + (-e_0)^2 \frac{k^2}{a_0} k_{\mu} D^{\mu\mu'}(k) D^{\nu\nu'}(\ell) G_{\mu\nu'}(p',p;k,\ell) \bar{\delta}(p'-k-p-\ell) = 0 . \quad (67) \end{aligned}$$

The same manipulations we performed on (61) lead us from (67) to the following identity

$$k^{\mu} G_{\mu\nu}(p',p;k,\ell) = G_{\nu}(p',p+k;p'-p-k) - G_{\nu}(p'-k,p;p'-p-k) . \quad (68)$$

By induction we can, without any difficulty, derive a general formula for the gauge identities:

$$\begin{aligned} k^{\mu} G_{\mu\nu_1 \dots \nu_n}(p',p;k,\ell_1, \dots, \ell_n) = G_{\nu_1 \dots \nu_n}(p',p+k;\ell_1, \dots, \ell_n) \\ - G_{\nu_1 \dots \nu_n}(p'-k,p;\ell_1, \dots, \ell_n) . \quad (69) \end{aligned}$$

As required by momentum conservation, the momenta appearing in (69) satisfy the relationship

$$p' = p + k + \ell_1 + \dots + \ell_n . \quad (70)$$

(69) is an open system of identities equating the divergences of the $(n+1)$ -point photon amputated Green functions to the n -point ones. These identities are constraints on the Green functions, which should in principle be respected by any manipulation we perform on or any results we obtain from our theory.

The quantities we wrote down above are all divergent and therefore the identities can only be regarded as having formal meaning: renormalization is obviously necessary. Applying the argument we gave in last section when we dealt with the renormalization of the Dyson-Schwinger equations, we can see that forms of all the identities remain unchanged after renormalization. We quote them below for future use.

$$\frac{k^2}{a_R} k_\mu D_R^{\mu\nu}(k) + k^\nu = 0 , \quad a_R = Z_A^{-1} a_0 , \quad (71)$$

$$k^\mu G_{R\mu}(p', p; k) = S_R(p) - S_R(p') \quad (72)$$

$$\text{and } k^\mu G_{R\mu\nu_1 \dots \nu_n}(p', p; k, \ell_1, \dots, \ell_n) = G_{R\nu_1 \dots \nu_n}(p', p+k; \ell_1, \dots, \ell_n) \\ - G_{R\nu_1 \dots \nu_n}(p'-k, p; \ell_1, \dots, \ell_n) . \quad (73)$$

Here again we use "R" to distinguish renormalized quantities from the corresponding unrenormalized ones.

We will now provide the promised derivation of the Ward-Takahashi identities in terms of the one-particle-irreducible vertex generating functional. This functional is defined by

$$\Gamma[\bar{\psi}, \psi, A_\mu] = W[\bar{J}, J, J_\mu] + \int d^4k [\bar{J}(k)\psi(k) + \bar{\psi}(k)J(k) + J_\mu(k)A^\mu(-k)] . \quad (74)$$

Where $\bar{\psi}$, ψ and A_μ are no longer the field operators themselves but their vacuum expectation values (though we use the same symbols for both):

$$\begin{aligned}
A_\mu(-k) &= -\delta W[\bar{J}, J, J_\mu] / \delta J^\mu(k) \\
\psi(k) &= -\delta W[\bar{J}, J, J_\mu] / \delta \bar{J}(k) \\
\bar{\psi}(k) &= -\delta W[\bar{J}, J, J_\mu] / \delta J(k) .
\end{aligned} \tag{75}$$

The inverse of (75) can be obtained from (74) as,

$$\begin{aligned}
J_\mu(k) &= \delta \Gamma[\bar{\psi}, \psi, A_\mu] / \delta A_\mu(-k) \\
J(k) &= \delta \Gamma[\bar{\psi}, \psi, A_\mu] / \delta \bar{\psi}(k) \\
\bar{J}(k) &= \delta \Gamma[\bar{\psi}, \psi, A_\mu] / \delta \psi(k) .
\end{aligned} \tag{76}$$

Inserting both (75) and (76) into (57) establishes a new version of that equation

$$\begin{aligned}
e_0 \int d^4t \left\{ -\frac{\delta \Gamma[\bar{\psi}, \psi, A_\mu]}{\delta \psi(t)} \psi(t-k) + \bar{\psi}(t+k) \frac{\delta \Gamma[\bar{\psi}, \psi, A_\mu]}{\delta \bar{\psi}(t)} \right\} - \frac{k^2}{a_0} k_\mu A^\mu(-k) \\
-k^\mu \frac{\delta \Gamma[\bar{\psi}, \psi, A_\mu]}{\delta A^\mu(k)} = 0 .
\end{aligned} \tag{77}$$

If we differentiate (77) with respect to $\bar{\psi}(p')$ from the left and $\psi(p)$ from the right, then set both $\bar{\psi}$ and ψ to zero, we obtain

$$e_0 \left\{ -\frac{\delta^2 \Gamma[0, 0, A_\mu]}{\delta \bar{\psi}(p') \delta \psi(p+k)} + \frac{\delta^2 \Gamma[0, 0, A_\mu]}{\delta \bar{\psi}(p'-k) \delta \psi(p)} \right\} - k^\mu \frac{\delta^3 \Gamma[0, 0, A_\mu]}{\delta \bar{\psi}(p') \delta \psi(p) \delta A_\mu(k)} = 0 . \tag{78}$$

Using the definition of the one-particle irreducible amplitude $\Gamma_{(n)}$:

$$\begin{aligned}
&\bar{\delta}(p'-p-k_1-k_2-\dots-k_n) \Gamma_{\mu_1 \mu_2 \dots \mu_n}(p', p; k_1, k_2, \dots, k_n) \\
&= (-e_0)^n \cdot \delta^{(2+n)} \Gamma[0, 0, 0] / \delta \bar{\psi}(p') \delta \psi(p) \delta A^{\mu_1}(k_1) \delta A^{\mu_2}(k_2) \dots \delta A^{\mu_n}(k_n) \tag{79}
\end{aligned}$$

and the well-known relationship

$$\frac{\delta^2 \Gamma[0, 0, 0]}{\delta \bar{\psi}(p') \delta \psi(p)} = S^{-1}(p) \bar{\delta}(p'-p) , \tag{80}$$

we can easily work out the Ward-Takahashi identities for the one-particle-irreducible vertices by calculating the vanishing A_μ limits of (78)

itself and the functional derivatives of (78) with respect to A_μ . A general formula may be obtained by induction

$$k^\mu \Gamma_{R\mu\nu_1\dots\nu_n}(p', p; k, \ell_1, \dots, \ell_n) = \Gamma_{R\nu_1\dots\nu_n}(p', p+k; \ell_1, \dots, \ell_n) - \Gamma_{R\nu_1\dots\nu_n}(p'-k, p; \ell_1, \dots, \ell_n), \quad (81)$$

with the momenta still constrained by the relationship (70). One thing worth mentioning is that for the three-point Green functions we have the relation (63), but similar equations do not exist for the more-than-three-point Green functions in general as they contain one-particle reducible pieces.

For scalar electrodynamics the gauge identities are easier to obtain due to the absence of Grassmann quantities in the theory. All the identities acquire exactly the same forms as those of spinor electrodynamics but with the Green functions replaced by the corresponding scalar electrodynamics ones. For example, the identity relating the divergence of the three-point Green's function to the scalar propagators reads

$$k^\mu G_{R\mu}(p', p; k) = \Delta_R(p) - \Delta_R(p'). \quad (82)$$

Consistency of Dyson-Schwinger equations and gauge identities

Before closing this section we consider the problem as to whether the Ward-Green-Takahashi identities are consistent with the Dyson-Schwinger equations. The two must be consistent, if the theory, governed by the one dynamical Lagrangian (1), is not to lead to absurdities. Specifically we will look at the problem of consistency in covariant gauges and we elect to work with the unrenormalized Dyson-Schwinger equations and gauge identities. (The method we will utilize here can also be applied in dealing with the renormalized version of the problem,

though it is a bit more cumbersome in this case because of the presence of counterterms.)

First of all we multiply the left-hand-side of equation (27) by $\frac{q^2}{a_0} q^\nu$ then apply identities (58), (60) and (66) to it. This results in

$$\begin{aligned}
 & e_0 \left\{ \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p'-q) \delta J(p)} - \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p+q)} \right\} (\not{p} - m_0) \\
 & + e_0 \int \bar{d}^4 k \left\{ -ie_0 \left[\frac{\delta^3 W[0,0,J_\mu]}{\delta \bar{J}(p'-q) \delta J(p-k) \delta J^\mu(k)} - \frac{\delta^3 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p+q) \delta J^\mu(k)} \right] \right. \\
 & + e_0 \frac{\delta W[0,0,J_\mu]}{\delta J^\mu(k)} \left[\frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p'-q) \delta J(p-k)} - \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p+q-k)} \right] \\
 & \left. - q_\mu \bar{\delta}(k+q) \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p-k)} \right\} \gamma^\mu \quad (83)
 \end{aligned}$$

Now rearrange the terms in (83) into the following order

$$\begin{aligned}
 & e_0 \left\{ \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p'-q) \delta J(p)} (\not{p} - m_0) - e_0 \int \bar{d}^4 k \left[i \frac{\delta^3 W[0,0,J_\mu]}{\delta \bar{J}(p'-q) \delta J(p-k) \delta J^\mu(k)} \right. \right. \\
 & \quad \left. \left. - \frac{\delta W[0,0,J_\mu]}{\delta J^\mu(k)} \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p'-q) \delta J(p-k)} \right] \gamma^\mu \right\} \\
 & - e_0 \left\{ \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p+q)} (\not{p} + \not{q} - m_0) - e_0 \int \bar{d}^4 k \left[i \frac{\delta^3 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p-k+q) \delta J^\mu(k)} \right. \right. \\
 & \quad \left. \left. - \frac{\delta W[0,0,J_\mu]}{\delta J^\mu(k)} \frac{\delta^2 W[0,0,J_\mu]}{\delta \bar{J}(p') \delta J(p+q-k)} \right] \gamma^\mu \right\} . \quad (84)
 \end{aligned}$$

Equation (24) tells us that (84) identically vanishes. We thereby satisfy ourselves that the Ward-Takahashi identities do not violate the Dyson-Schwinger equations. In exactly the same way we can also prove the consistency of the conjugate equations and identities.

The interesting point of the proof we have given above lies in the fact that it suggests a method for truncating the Dyson-Schwinger equations, while at the same time respecting the Ward-Green-Takahashi identities. The method consists in truncating the equations according to the superficial appearance of the coupling constant e^2 . We will adopt this method in the next section.

II.3 SPECTRAL REPRESENTATIONS OF CHARGED PARTICLE PROPAGATORS IN COVARIANT GAUGES

Although relativistic quantum field theory is so intractable that any computation carried out using it relies on perturbation theory or some other approximation schemes, certain exact properties of field theory quantities are known. In the last two sections we studied the field equations and the gauge identities merely by analysing the Lagrangian governing the dynamics. Here we will consider another exact feature of relativistic field theories - the spectral representations of propagators. The existence of these representations can be deduced using only assumptions basic to quantum field theories. They were obtained separately by Källén, Wightman and Lehmann [32, 33] in the early 50's and have become so well-known that their detailed derivations are not necessary. We will merely list the relevant results below.

The simplest case is a scalar field interacting with others. Its renormalized propagator can be represented as a weighted sum over meson mass distributions of tree graphs. That is, in a covariant gauge,

$$\Delta(p) = \int_m^\infty dW^2 \frac{\rho(W^2)}{p^2 - W^2 + i0^+} . \quad (85)$$

It is important to distinguish $\Delta(p)$, the renormalized propagator from the unrenormalized propagator. We have dropped the "R" suffix for

convenience, and hope that no confusion will occur as a result.

In (85), m is the physical mass of the scalar particle and $\rho(W^2)$ is a real scalar function of the variable W^2 and the gauge parameter a , but not the momentum p . There is a cut in the p^2 -plane along the real axis from m^2 to $+\infty$, and the discontinuity across this cut defines the spectral function $\rho(W^2)$. Let us apply the identity

$$\frac{1}{x+i0^+} = \frac{P}{x} - i\pi\delta(x) \quad (86)$$

to (85) and take its imaginary part, we arrive at

$$\rho(p^2) = -\frac{1}{\pi} \text{Im}\Delta(p) \quad (87)$$

The combination of (87) and (85) gives an equivalent representation of $\Delta(p)$

$$\Delta(p) = \frac{1}{\pi} \int_m^\infty dW^2 \frac{\text{Im}\Delta(W)}{W^2 - p^2 - i0^+} \quad (88)$$

This is nothing else but the dispersion relation of $\Delta(p)$ in the complex p^2 -plane.

Two sum rules concerning the spectral function can be extracted,

$$Z_\phi^{-1} = \int_m^\infty dW^2 \rho(W^2), \quad Z_\phi^{-1} m_0^2 = \int_m^\infty dW^2 W^2 \rho(W^2) \quad (89)$$

by studying the ultra-violet properties of the propagator ($p \rightarrow \infty$ in (85)).

Due to the fact that the electron field is represented by a spinor, the renormalized propagator has a rather more complicated integral representation. Instead of having a single spectral function as in (85), two independent functions exist

$$S(p) = \int_m^\infty dW^2 \frac{\not{p}\rho_1(W^2) + m\rho_2(W^2)}{p^2 - W^2 + i0^+} \quad (90)$$

with m the physical mass of an electron. To simplify (90) we utilize the simple but important formula

$$\frac{1}{p^2 - W^2 + i0^+} = \frac{1}{2W} \left[\frac{1}{p - W + i0^+} - \frac{1}{p + W - i0^+} \right] \quad (91)$$

and change the integration region to C : $(-\infty, -m)$ and $(+m, +\infty)$. A neater expression is obtained, namely

$$S(p) = \left(\int_{-\infty}^{-m} + \int_m^{+\infty} \right) dW \frac{\rho(W)}{p - W + i\epsilon(W)0^+} \quad (92)$$

$\rho(W)$, which is independent of the variable p , is related to the two functions of (90) through

$$\rho(W) = \epsilon(W) [W\rho_1(W) + m\rho_2(W)] \quad (93)$$

Where $\epsilon(W)$ is the usual sign function

$$\epsilon(W) = \begin{cases} +1 & W > 0 \\ -1 & W < 0 \end{cases} \quad (94)$$

This makes (92) different from the usual Feynman " $i0^+$ " description.

By considering the asymptotic behaviour of $S(p)$, the following sum rules can be established

$$Z_\psi^{-1} = \int dW \rho(W), \quad Z_\psi^{-1} m_0 = \int dW W\rho(W) \quad (95)$$

where the integration region is still C .

To consider (92) from the viewpoint of dispersion relation, we replace the variable p by a scalar M , then take its imaginary part, we arrive at

$$\rho(M) = - \frac{\epsilon(W)}{\pi} \text{Im} S(M) \quad (96)$$

This enables us to express S through a usual dispersion relation in the complex M -plane,

$$\text{i.e.} \quad S(M) = \frac{1}{\pi} \int_C dW \frac{\epsilon(W) \text{Im}S(M)}{W-M-i\epsilon(W)0^+} \quad (97)$$

We will not analyse the gauge properties of the charged particle propagators in this thesis, but we should mention that both $\Delta(p)$ and $S(p)$ obey the Landau-Khalatnikov-Zumino identity under gauge transformations, and this in turn imposes definite gauge transformation rules for the spectral functions. The interested readers are referred to the relevant literature [29, 30, 31] [36] [40, 41].

II.4 NONCOVARIANT GAUGES

We have exclusively dealt with covariant gauges so far, where a Lorentz invariant gauge fixing term $-\frac{1}{2a_0} (\partial_\mu A^\mu)^2$ was introduced into the Lagrangian through the standard Fadeev-Popov procedure [74]. Though the concrete expression of the gauge fixing term did not influence the derivation of some of the Dyson-Schwinger equations, it did play a minor role in deriving the Ward-Green-Takahashi identities as well as the photon propagator Dyson-Schwinger equations. In this section we will consider what changes are brought upon the forms of the field equations and gauge identities by altering the gauge fixing term.

Instead of (7) suppose we choose

$$F(x) = n^\mu A_\mu(x) \quad (98)$$

with n an arbitrary external vector. The determinant

$$\det\left(\frac{\delta F}{\delta \Lambda}\right) = \det(n^\mu \partial_\mu) \quad (99)$$

is still independent of integration variables and the external sources, hence can be absorbed in the overall normalization constant. Here the gauge fixing Lagrangian is

$$\mathcal{L}_g = -\frac{1}{2a_0} (n^\mu A_\mu)^2 \quad (100)$$

An important feature of (100) is that a scale change of n can be compensated by rescaling a_0 , as it is an arbitrary parameter. Hence the n vector can be chosen such that

$$n^2 = +1, 0, -1, \quad (101)$$

without losing generality. Therefore three kinds of gauges are specified by different n , namely, the time-like axial gauge, light-cone gauge and space-like axial gauge, corresponding to n^2 equal to $+1$, 0 and -1 respectively. In the limit that a_0 vanishes, the gauge condition

$$n^\mu A_\mu(x) = 0 \quad (102)$$

is recovered. With the gauge choice (100) we can write out the momentum space vacuum functional as usual,

$$Z[\bar{J}, J, J_\mu] = \int [d\psi d\bar{\psi} dA_\mu] \exp(iE), \quad (103)$$

with E the effective action given by

$$E = \int d^4k \left\{ -\frac{1}{4} F_{\mu\nu}(k) F^{\mu\nu}(-k) + \bar{\psi}(k)(\not{k} - m_0)\psi(k) - e_0 \int d^4k' \bar{\psi}(k) A(k-k') \psi(k') \right. \\ \left. - \frac{1}{2a_0} [n_\mu A^\mu(k) n_\nu A^\nu(-k)] - \bar{J}(k)\psi(k) - \bar{\psi}(k)J(k) - J_\mu(k)A^\mu(-k) \right\}, \quad (104)$$

which is different from (11) only in the gauge fixing term. So far as Dyson-Schwinger equations are concerned all the arguments and results of Section 1 are still valid here with one minor change to the photon propagator equations. The inverse bare photon propagator is no longer (39) but given by

$$D_{\mu\nu}^{(0)-1}(k) = (-k^2 g_{\mu\nu} + k_\mu k_\nu) - \frac{1}{a_0} n_\mu n_\nu. \quad (105)$$

Next we turn to the derivation of Ward-Green-Takahashi identities in axial gauges. The gauge transformed action now is

$$E^\Lambda = E - \int d^4k \left\{ -ie_0 \bar{J}(k) \int d^4t \Lambda(t) \psi(k-t) + ie_0 \int d^4t \Lambda(t) \bar{\psi}(k+t) J(k) - ik_\mu J^\mu(-k) \Lambda(k) \right. \\ \left. - \frac{i}{a_0} n \cdot k n^\mu A_\mu(-k) \Lambda(k) \right\} . \quad (106)$$

This, together with the gauge invariance of the vacuum functional, generates the basic equation

$$e_0 \int d^4t \left\{ \bar{J}(t) \frac{\delta W[\bar{J}, J, J_\mu]}{\delta \bar{J}(t-k)} - \frac{\delta W[\bar{J}, J, J_\mu]}{\delta J(t+k)} J(t) \right\} + \frac{n \cdot k}{a_0} n_\mu \frac{\delta W[\bar{J}, J, J_\mu]}{J_\mu(k)} \\ - k^\mu J_\mu(-k) = 0 . \quad (107)$$

Following the strategy of Section 2 we can obtain the photon propagator gauge identity

$$\frac{n \cdot k}{a} n_\mu D^{\mu\nu}(k) + k^\nu = 0 , \quad a = Z_A^{-1} \cdot a_0 \quad (108)$$

with a and $D_{\mu\nu}$ renormalized, and

$$e_0 \left\{ \frac{\delta^2 W[0, 0, J_\mu]}{\delta \bar{J}(p'-k) \delta J(p)} - \frac{\delta^2 W[0, 0, J_\mu]}{\delta \bar{J}(p') \delta J(p+k)} \right\} + \frac{n \cdot k}{a_0} n_\mu \frac{\delta^3 W[0, 0, J_\mu]}{\delta \bar{J}(p') \delta J(p) \delta J_\mu(k)} = 0 . \quad (109)$$

The vanishing external source limit of the n th order derivative of (109) with respect to J produces the general formula,

$$k^\mu G_{\mu\nu_1 \dots \nu_n}(p', p; k, \ell_1, \dots, \ell_n) = G_{\nu_1 \dots \nu_n}(p', p+k; \ell_1, \dots, \ell_n) \\ - G_{\nu_1 \dots \nu_n}(p'-k, p; \ell_1, \dots, \ell_n) , \quad (110)$$

with the application of (108). (110) agrees with (73) exactly, so we conclude that the electrodynamics identities retain their forms in all gauges.

The spectral representations of propagators become very cumbersome in non-covariant gauges. The commonly accepted and widely applied spectral representations in the axial gauge ($n^2 \neq 0$, $a=0$) are [38, 39]

$$\Delta(p) = \int_m^\infty dW^2 \frac{\rho(W^2; p.n)}{p^2 - W^2 + i0^+}, \quad (111)$$

$$S(p) = \int_m^\infty dW^2 [\not{p}\rho_1(W^2; p.n) + m\rho_2(W^2; p.n) + \frac{\not{p}W^2}{p.n} \rho_3(W^2; p.n)] / (p^2 - W^2 + i0^+), \quad (112)$$

though rigorous proofs of these representations do not exist [34].

Because of the n scale invariance of the theory, the spectral functions can only depend on $p.n$ through the gauge parameter

$$\gamma = \frac{p^2 \cdot n^2}{(p.n)^2}. \quad (113)$$

In the light-cone gauge, γ vanishes, so the representations (111) and (112) reduce to

$$\Delta(p) = \int_m^\infty dW^2 \frac{\rho(W^2)}{p^2 - W^2 + i0^+} \quad (114)$$

$$S(p) = \int_m^\infty dW^2 [\not{p}\rho_1(W) + m\rho_2(W^2) + \frac{\not{p}W^2}{p.n} \rho_3(W^2)] / (p^2 - W^2 + i0^+) \quad (115)$$

with all the spectral functions being n independent [60].

III. THE GAUGE TECHNIQUE IN SPINOR ELECTRODYNAMICS

Salam [17] introduced the gauge technique in 1963 with the aim of renormalizing vector electrodynamics. The method consisted in making an ansatz, which satisfied the gauge identity, for the three-point vertex function in the form of a weighted sum with the same spectral function as the one appearing in the spectral representation of the inverse propagator of the charged particle, and then combining this ansatz with two-particle unitarity to determine the spectral function itself. The method possessed the virtue that the gauge identity was respected at any stage of iteration in contrast to other approximate methods, where the gauge identity was often violated.

However, the gauge technique had its own limitations. The approximate vertex function was inherently non-unique since its transversal part could not possibly be determined by the gauge identity; and when one went to higher orders of iteration, the method became too involved to be amenable. These problems hindered the development of the gauge technique until late 70's when the paper of Delbourgo and West [35] appeared and a revised version of the method was started. It has since then been widely used in studies of gauge theories, especially, quantum electrodynamics, although the question of transverse ambiguities has remained unsolved till very recently.

We will give a short review of the gauge technique in spinor electrodynamics in Section 1, then in Sections 2 and 3, we deal with the main problem of this chapter: to construct a new vertex function by including certain transversality and apply this vertex to solve the Dyson-Schwinger equation of the electron propagator. In the final section we consider the photon vacuum polarization.

III.1 INTRODUCTION TO THE LOWEST ORDER GAUGE TECHNIQUE

In this section we will review some aspects of the lowest order gauge technique and its applications in spinor electrodynamics. We start from Reference [35]. In this paper, the authors abandoned the two-particle unitarity as the starting point of the analysis. Rather they studied the Dyson-Schwinger equation of the spinor particle propagator utilizing an ansatz for the three-point photon amputated Green function. By discarding the photon dressing, which did not bear much importance to the lowest order, they obtained a linear integral equation for the spectral function. Although an important modification was made to the gauge technique, the essence of Salam's original ideas remained: the approximate three-point amplitude they applied was deduced from the Ward-Green-Takahashi identity; therefore all the advantages of the original gauge technique were maintained.

Starting from the identity (II-72), the authors manipulated it into the form

$$k^\mu G_\mu(p, p-k; k) = \int dW \rho(W) \frac{1}{\not{p}-W} \not{k} \frac{1}{\not{p}-\not{k}-W}, \quad (1)$$

with the application of the spectral representation of the fermion propagator, (II-92). In (1) the integration region is $(-\infty, -m)$ and $(+m, +\infty)$ and the ' $i\epsilon(W)0^+$ ' description should be understood.

By imposing the restriction to the solution of (1) that when one went to the mass shell of the spinor lines (by picking out the pole terms, i.e. replacing $\rho(W)$ with $\delta(W-m)$) it should be possible to obtain the exact Born term, they found the possible solution of (1),

$$G_\mu^{(0)}(p, p-k; k) = \int dW \frac{1}{\not{p}-W} \gamma_\mu \frac{1}{\not{p}-\not{k}-W}. \quad (2)$$

In analogy to (2) a corresponding expression for $G_{\nu\mu}$ was also obtained

$$G_{\nu\mu}^{(0)}(p', p; -k', k) = - \int dW \rho(W) \frac{1}{\not{p}' - W} [\gamma_\nu \frac{1}{(\not{p}' + \not{k}') - W} \gamma_\mu + \gamma_\mu \frac{1}{(\not{p}' - \not{k}) - W} \gamma_\nu] \frac{1}{\not{p} - W} , \quad (3)$$

through the proper Ward-Green-Takahashi identity given by (II-73).

As we can easily see, although the above restriction is very strong, it does not completely wipe out the ambiguity in the solutions of the gauge identities as arbitrary transverse pieces can be added to (2) and (3) as long as they vanish on the fermion mass shell. In fact this ambiguity can only be eliminated after a successful determination of the transverse vertices.

The next step was to apply (2) to the Dyson-Schwinger equation

$$Z_\psi^{-1} = S(p)(\not{p} - m_0) - ie^2 \int d^4k G_\mu(p, p-k; k) \gamma_\nu D^{\mu\nu}(k) . \quad (4)$$

To the lowest order the authors neglected the photon dressing by replacing the full renormalized photon propagator with the bare one

$$D_{\mu\nu}^{(0)}(k) = - \frac{1}{k^2} [g_{\mu\nu} - (1-a) \frac{k_\mu k_\nu}{k^2}] . \quad (5)$$

In that way (4) was simplified to a linear integral equation for the spectral function $\rho(W)$:

$$Z_\psi^{-1} = \int dW \rho(W) \frac{1}{\not{p} - W} [\not{p} - m_0 + \Sigma^{(0)}(p, W)] , \quad (6)$$

with $\Sigma^{(0)}(p, W)$, the lowest order mass correction to a fermion of mass W , given by

$$\Sigma^{(0)}(p, W) = -ie^2 \int d^4k \gamma_\mu \frac{1}{\not{p} - \not{k} - W} \gamma_\nu D^{(0)\mu\nu}(k) . \quad (7)$$

By recalling the sum rules (II-95), and decomposing $\rho(W)$ to

$$\rho(W) = \delta(W-m) + \sigma(W) , \quad (8)$$

(6) was manipulated into

$$\frac{\Sigma^{(0)}(W, m)}{W - m} + \int dW' [W' - m_0 + \Sigma^{(0)}(W, W')] \frac{\sigma(W')}{W - W'} = 0 \quad (9)$$

after the matrix was replaced by the scalar quantity W . Upon taking the imaginary part of (9), and replacing $m_0 - \Sigma^{(0)}(W, W)$ by m , the renormalized mass (here the divergence arised from $m_0 - \Sigma^{(0)}(W, W)$ was neglected due to the fact that it was of higher order in e^2), the authors arrived at the following linear integral equation for the spectral function $\sigma(W)$,

$$\epsilon(W)(W - m)\sigma(W) = \frac{\text{Im}\Sigma^{(0)}(W, m)}{\pi(W - m)} + \frac{1}{\pi} \int dW' \sigma(W') \frac{\text{Im}\Sigma^{(0)}(W, W')}{W - W'} \quad (10)$$

The absorptive part of the mass correction $\Sigma^{(0)}(p, W)$ was worked out explicitly; it reads

$$\text{Im}\Sigma^{(0)}(p, W) = \pi \left(\frac{e}{4\pi}\right)^2 \cdot \frac{p^2 - W^2}{p^3} [a(p^2 + W^2) - (a+3)pW] \theta(p^2 - W^2). \quad (11)$$

Due to its complexity the authors could only manage to solve (10) in the Landau gauge, $a=0$, where it reduced to

$$\frac{(\omega - 1)S(\omega)}{\xi} = (\omega + 1) + \left(\int_1^{\epsilon(\omega)\omega} - \int_{-\epsilon(\omega)\omega}^{-1} \right) d\omega' S(\omega') \left(1 + \frac{\omega}{\omega'}\right), \quad (12)$$

with the dimensionless quantities ω , $S(\omega)$ and ξ defined, respectively, by

$$\omega = \frac{W}{m}, \quad \xi = -3\left(\frac{e}{4\pi}\right)^2, \quad S(\omega) = \epsilon(\omega)\omega \cdot W\sigma(W) \quad (13)$$

(12) was further simplified, with the decomposition,

$$S(\omega) = \omega s_1(\omega^2) + s_2(\omega^2) \quad (14)$$

to two coupled integral equations in terms of s_1 and s_2 ,

$$\begin{aligned} [s_2(\omega^2) - s_1(\omega^2)]/\xi &= \int_1^{\omega^2} d\omega'^2 s_2(\omega'^2)/\omega'^2 + 1 \\ [\omega^2 s_1(\omega^2) - s_2(\omega^2)]/\xi &= \int_1^{\omega^2} d\omega'^2 s_1(\omega'^2) + 1. \end{aligned} \quad (15)$$

When (15) was converted to differential equations, two hypergeometric equations were recognized, of which the solutions satisfying proper boundary conditions were

$$s_1(z) = \frac{2\xi}{z-1} \left(\frac{z-1}{\frac{\mu^2}{m^2}} \right)^{2\xi} F(\xi, \xi; 2\xi; 1-z)$$

$$s_2(z) = \frac{2\xi z}{z-1} \left(\frac{z-1}{\frac{\mu^2}{m^2}} \right)^{2\xi} F(\xi, \xi+1; 2\xi; 1-z) .$$

This in turn gave out the $\sigma(W)$ function in its original variables

$$\sigma(W) = \epsilon(W) \theta(W^2 - m^2) \frac{2\xi}{W} \left(\frac{W^2 - m^2}{\frac{\mu^2}{m^2}} \right)^{2\xi} \frac{m^2}{W^2 - m^2} \left\{ F(\xi, \xi; 2\xi; 1 - \frac{W^2}{m^2}) \right. \\ \left. + \frac{W}{m} F(\xi, \xi+1; 2\xi; 1 - \frac{W^2}{m^2}) \right\} \quad (16)$$

incorporating the infrared cut-off μ^2 .

In fact the decomposition (8) is not correct in infrared region, $W \rightarrow m$, except for the Yennie gauge, $a=3$. Being aware of this fact the authors returned to the original $\rho(W)$ and directly studied the integral equation

$$\epsilon(W) \rho(W) (W-m) = \frac{1}{\pi} \int dW' \rho(W') \frac{\text{Im}\Sigma^{(0)}(W, W')}{W-W'} . \quad (17)$$

By approximating the Dyson-Schwinger equation for S^{-1} in a manner consistent with perturbation theory, Atkinson and Slim [76] obtained an equation similar to (17),

$$\epsilon(W) \rho(W) (W-m) \left\{ 1 + 3 \left(\frac{\hat{e}}{4\pi} \right)^2 \cdot \left[\frac{W}{W-m} \ln \frac{W^2}{m^2} - 2 \right] \right\} = \frac{1}{\pi} \int dw' \rho(W') \frac{\text{Im}\Sigma^{(0)}(W, W')}{W-W'} . \quad (17')$$

As we will see in section 3, when we include the transverse vertex contributions and properly renormalize our gauge technique equation, terms similar to those in the curled brackets of (17') will arise. However, here they can be neglected and (17') agrees completely with (17).

Now we follow Delbourgo and West's method [37] to solve (17) in

the infrared. Let $W \rightarrow m$, (17) reduces to

$$\rho(W)(W-m) = 2\left(\frac{e}{4\pi}\right)^2 (a-3) \cdot \int_m^W dW' \rho(W') , \quad (18)$$

its solution can be written out straightforwardly

$$\rho(W) = c \left(\frac{1}{W-m}\right)^{1-2\eta(a-3)} , \quad (19)$$

with $\eta = \left(\frac{e}{4\pi}\right)^2$ here and throughout, and c an arbitrary constant. And (19) in turn gives out the infrared behaviour of the fermion propagator

$$S(p) = c \left(\frac{1}{p-m}\right)^{1-2\eta(a-3)} .$$

This result was previously obtained by other people [77] [78] by summing up perturbative terms, but as Delbourgo and West claimed the above is the simplest derivation of this famous result.

In the ultra-violet limit, Atkinson and Slim [76] analysed (17) by decomposing the spectral function into an even and odd part,

$$\rho^\pm(W) = \rho(W) \pm \rho(-W) , \quad (20)$$

and assuming the solutions

$$\rho^+(W) \sim W^\gamma , \quad \rho^-(W)/\rho^+(W) \sim 0 , \quad W \rightarrow +\infty . \quad (21)$$

They found that a possible solution for small coupling constant was

$$\gamma \sim -1 + 2\eta a$$

that is

$$\rho^+(W) \sim W^{-1+2\eta a} . \quad (22)$$

Fixing on the Landau gauge, $a=0$, Delbourgo and West [37] also solved (17) in all momentum regions. Their solution is

$$\rho(W) = \frac{6\eta R(\eta)}{\left(\frac{W^2}{m^2} - 1\right)^{1+6\eta}} \left[\frac{1}{W} F(-3\eta, -3\eta; -6\eta; 1 - \frac{W^2}{m^2}) + \frac{1}{m} F(-3\eta, 1-3\eta; -6\eta; 1 - \frac{W^2}{m^2}) \right] , \quad (23)$$

where $R(n)$ is an arbitrary constant (independent of W) normalized to give unity in the free field limit.

A more thorough study of equation (17) can be found in [36], [41] and [79]. In [36] Slim solved (17) in all momentum regions and all gauges employing Mellin transformations, and the solution turned out to be Meijer G functions [80]. The gauge property of this solution was analyzed in [36] and [41]. The authors found that the solution respected the Zumino identity in asymptopia, but not at intermediate energies. However, when the solution was expanded into perturbative series, the discrepancy disappeared to $O(e^2)$ even at intermediate momenta.

III.2 THE INCLUSION OF TRANSVERSE VERTICES⁺

Although the lowest order gauge technique produces a reasonable spectral function, especially in infrared and ultra-violet regions (where the function qualifies as gauge covariant); in intermediate momentum region it loses gauge covariance due to the neglect of the transverse contributions. However, whether the transverse vertex does restore gauge covariance still remains unknown at the level of equation (2) and it can only be answered by explicit analysis of the transverse contributions. In any case without taking the transverse contributions into account, the gauge technique can not possibly give any information about problems concerning form factors of the electron. In two dimensions, where the vector particle acquires a mass, the need for transverse vertex is even more pressing [43, 44]. In this section we will develop a new method for determining the transverse vertex, but before doing so, we give a short review of King's work [53].

⁺ Section III.2 and III.3 are based on Reference [55].

Very recently, two notable attempts at incorporating transverse amplitudes were made by King [53] and Parker [54]. Confining himself to scalar electrodynamics, Parker established a method to include the first order corrections to the vertex. We will discuss his work in the next chapter. As we will see, our method is closely related to his.

King's approach instead largely relied on perturbation theory. In the absence of fermionic loops, King was only interested in the leading logarithmic effects of the electron self-energy in asymptopia. His longitudinal vertex was obtained from the ansatz of $G_\mu^{(0)}$ [35] by manipulations of Dirac algebras, and then he was able to introduce a transverse vertex, which was of the order e^2 and regularized to meet the requirements that, as both the fermionic lines going on mass-shell, it should vanish. Though his vertex might agree with the true vertex function in the asymptopia, there was no guarantee that it would be close to the real vertex at intermediate momenta.

The Inclusion of Transverse Amplitudes

We will instead attack the problem in a completely different way. Restating the Dyson-Schwinger equations for the three-point Green function,

$$\text{i.e. } G_\mu(p', p; p'-p)(\not{p}-m_0) = S(p')\gamma_\mu - ie^2 \int \bar{d}^4 k G_{\mu\nu}(p', p-k; p'-p, k) \gamma_\rho D^{\nu\rho}(k) \quad (24a)$$

$$(\not{p}'-m_0)G_\mu(p', p; p'-p) = \gamma_\mu S(p) - ie^2 \int \bar{d}^4 k \gamma_\rho G_{\mu\nu}(p'+k, p; p'-p, k) D^{\nu\rho}(k), \quad (24b)$$

and subtracting (24a) from (24b), we arrive at

$$\not{p}' G_\mu(p', p; p'-p) - G_\mu(p', p; p'-p) \not{p} = F_\mu(p', p) + e^2 H_\mu(p', p), \quad (25)$$

$$\text{with } F_\mu(p', p) = \gamma_\mu S(p) - S(p')\gamma_\mu, \quad (26)$$

$$H_\mu(p', p) = -i \int \bar{d}^4 k D^{\nu\rho}(k) [\gamma_\rho G_{\mu\nu}(p'+k, p; p'-p, k) - G_{\mu\nu}(p', p-k; p'-p, k) \gamma_\rho]. \quad (27)$$

(25) relates the three-point Green function to the four-point one, of which the spectral representation is unknown. Therefore an ansatz for $G_{\mu\nu}$ is necessary, in order for us to bring (25) into a closed form. From (25) we can work out G_μ in terms of the functions F_μ and H_μ . By multiplying (25) with \not{p}' to the left and \not{p} to the right, we obtain

$$\not{p}'^2 G_\mu(p', p; p' - p) - \not{p}' G_\mu(p', p; p' - p) \not{p} = \not{p}' [F_\mu(p', p) + e^2 H_\mu(p', p)] \quad (28a)$$

$$\not{p}' G_\mu(p', p; p' - p) \not{p} - G_\mu(p', p; p' - p) p^2 = [F_\mu(p', p) + e^2 H_\mu(p', p)] \not{p}. \quad (28b)$$

Adding (28a) and (28b) together and dividing the resultant equation by $\not{p}'^2 - p^2$ results in

$$G_\mu(p', p; p' - p) = [\not{p}' F_\mu(p', p) + F_\mu(p', p) \not{p}] / (\not{p}'^2 - p^2) + e^2 [\not{p}' H_\mu(p', p) + H_\mu(p', p) \not{p}] / (\not{p}'^2 - p^2). \quad (29)$$

Applying the covariant gauge spectral representation of the spinor propagator to the first piece, we can simplify (29) to

$$G_\mu(p', p; p' - p) = \int dW \frac{\rho(W)}{\not{p}' - W} \gamma_\mu \frac{1}{\not{p} - W} + e^2 [\not{p}' H_\mu(p', p) + H_\mu(p', p) \not{p}] / (\not{p}'^2 - p^2). \quad (30)$$

(30) is an exact equation, the first term of which on the right-hand-side is exactly the Delbourgo-West ansatz. This shows us a very important fact that, in the case of small coupling, the lowest order gauge technique can and does produce a satisfactory result in any momentum region, but for larger e^2 , this statement is no longer true. Only in certain momentum regions, such as the infrared, where the second piece of (30) does not contribute to the G_μ significantly, is the gauge technique expected to reveal true properties of the theory.

Now we manipulate the Dyson-Schwinger equations (II.46a) and (II.46b) as we did above. We obtain an exact relation among the three-, four- and five-point Green functions,

$$G_{\mu\nu}(p', p; \ell, p' - p - \ell) = \frac{\not{p}' M_{\mu\nu}(p', p, \ell) + M_{\mu\nu}(p', p, \ell) \not{p}}{p'^2 - p^2} + e^2 \frac{\not{p}' N_{\mu\nu}(p', p, \ell) + N_{\mu\nu}(p', p, \ell) \not{p}}{p'^2 - p^2} \quad (31)$$

with

$$M_{\mu\nu}(p', p, \ell) = [G_\mu(p', p' - \ell; \ell) \gamma_\nu + G_\nu(p', p + \ell; p' - p - \ell) \gamma_\mu] - [\gamma_\mu G_\nu(p' - \ell, p; p' - p - \ell) + \gamma_\nu G_\mu(p + \ell, p; \ell)] , \quad (32)$$

$$N_{\mu\nu}(p', p, \ell) = i \int \bar{d}^4 k [G_{\mu\nu\rho}(p', p - k; \ell, p' - p - \ell, k) \gamma_\rho - \gamma_\rho G_{\mu\nu\rho}(p' + k, p; \ell, p' - p - \ell, k)] D^{\rho\rho'}(k) . \quad (33)$$

By applying (30) to (31) and discarding the e^2 terms, which again is reasonable only when the coupling constant e^2 is small, (31) gives out the lowest order $G_{\mu\nu}$ as a weighted sum of the Born term

$$G_{\mu\nu}^{(0)}(p', p; \ell, p' - p - \ell) = - \int dW \frac{\rho(W)}{p - W} \left\{ \gamma_\mu \frac{1}{p' - \ell - W} \gamma_\nu + \gamma_\nu \frac{1}{p + \ell - W} \gamma_\mu \right\} \frac{1}{p - W} , \quad (34)$$

corresponding to the ansatz given by Delbourgo and West [35]. When (34) is inserted in (27), $H_\mu(p', p)$ reduces to its lowest order form:

$$H_\mu^{(0)}(p', p) = -i \int dW \rho(W) \int \bar{d}^4 k \frac{1}{p' - W} \left[\gamma_\nu \frac{1}{p' - k - W} \gamma_\mu \frac{1}{p - k - W} \gamma_\lambda \not{p} - \not{p}' \gamma_\lambda \frac{1}{p' - k - W} \gamma_\mu \frac{1}{p - k - W} \gamma_\nu \right] \frac{D^{\nu\lambda}(k)}{p - W} - i \int dW \rho(W) \int \bar{d}^4 k \frac{1}{p' - W} \left[\gamma_\mu \gamma_\nu \frac{1}{p - k - W} \gamma_\lambda - \gamma_\lambda \frac{1}{p' - k - W} \gamma_\nu \gamma_\mu \right] \frac{D^{\nu\lambda}(k)}{p - W} . \quad (35)$$

Let us define

$$\Lambda_\mu(p', p) = \frac{e^2}{p'^2 - p^2} [\not{p}' H_\mu(p', p) + H_\mu(p', p) \not{p}] . \quad (36)$$

By applying (35) to (36) and carrying out some tedious manipulations we arrive at

$$\Lambda_\mu(p', p) = ie^2 \int dW \rho(W) \int d^4 k \frac{1}{\not{p}' - \not{W}} \{ \gamma_\nu \frac{1}{\not{p}' - \not{k} - \not{W}} \gamma_\mu \frac{1}{\not{p} - \not{k} - \not{W}} \gamma_\lambda$$

$$- \left[\frac{\not{p}' \gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \gamma_\nu \frac{1}{\not{p} - \not{k} - \not{W}} \gamma_\lambda - \gamma_\nu \frac{1}{\not{p}' - \not{k} - \not{W}} \gamma_\lambda \frac{\not{p}' \gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \right] \frac{1}{\not{p} - \not{W}} D^{\nu\lambda}(k) .$$
(37)

Combining (30) with (37) gives an approximate $G_\mu(p', p)$ exact to order e^2 and containing a transverse part $\Lambda_\mu(p', p)$.

Properties of the Transverse Amplitude

At first sight (37) seems to contain a pole at $p'^2 = p^2$. In fact, this pole does not exist. By writing

$$\Sigma^{(2)}(p, W) = -ie^2 \int d^4 k \gamma_\mu \frac{1}{\not{p} - \not{k} - \not{W}} \gamma_\nu D^{\mu\nu}(k) ,$$
(38)

(37) can be expressed as

$$\Lambda_\mu(p', p) = ie^2 \int dW \rho(W) \int d^4 k \frac{1}{\not{p}' - \not{W}} \gamma_\nu \frac{1}{\not{p}' - \not{k} - \not{W}} \gamma_\mu \frac{1}{\not{p} - \not{k} - \not{W}} \gamma_\lambda \frac{1}{\not{p} - \not{W}} D^{\nu\lambda}(k)$$

$$+ \int dW \rho(W) \frac{1}{\not{p}' - \not{W}} \left[\frac{\not{p}' \gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \Sigma^{(2)}(p, W) - \Sigma^{(2)}(p', W) \frac{\not{p}' \gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \right] \frac{1}{\not{p} - \not{W}} .$$
(39)

Because

$$\Sigma^{(2)}(p, W) = A(p^2, W^2) \not{p} + B(p^2, W^2) W ,$$
(40)

with A and B scalar functions of the variables p^2 and W^2 , we have

$$\lim_{p'^2 \rightarrow p^2} \left[\frac{\not{p}' \gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \Sigma^{(2)}(p, W) - \Sigma^{(2)}(p', W) \frac{\not{p}' \gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \right]$$

$$= \lim_{p'^2 \rightarrow p^2} \left\{ \frac{A(p^2, W^2) - A(p'^2, W^2)}{p'^2 - p^2} \not{p}' \gamma_\mu \not{p} + \frac{p^2 A(p^2, W^2) - p'^2 A(p'^2, W^2)}{p'^2 - p^2} \gamma_\mu \right.$$

$$\left. + \frac{B(p^2, W^2) - B(p'^2, W^2)}{p'^2 - p^2} (\not{p}' \gamma_\mu + \gamma_\mu \not{p}) W \right\}$$

$$= - \frac{\partial A(p^2, W^2)}{\partial p^2} \not{p}' \gamma_\mu \not{p} - \frac{\partial p^2 A(p^2, W^2)}{\partial p^2} \gamma_\mu - \frac{\partial B(p^2, W^2)}{\partial p^2} (\not{p}' \gamma_\mu + \gamma_\mu \not{p}) W .$$
(41)

This is well-behaved as long as A and B are differentiable.

Recalling our claims at the end of Section II.2, we confidently expect that the G_μ obtained above satisfies Ward-Green-Takahashi identity (II-72), because we did truncate the Dyson-Schwinger equations in a manner consistent with perturbation theory. Put another way, the $\Lambda_\mu(p', p)$ should be purely transverse to $(p'-p)^\mu$. To check this out we multiply (39) with $(p'-p)^\mu$ and employ the simple but useful identity,

$$\frac{1}{\not{p}' - \not{k} - W} (\not{p}' - \not{p}) \frac{1}{\not{p} - \not{k} - W} = \frac{1}{\not{p} - \not{k} - W} - \frac{1}{\not{p}' - \not{k} - W} ,$$

and arrive at

$$\begin{aligned} \Lambda_\mu(p', p)(p'-p)^\mu &= \int dW \rho(W) \frac{1}{\not{p}' - W} [\Sigma^{(2)}(p, W) - \Sigma^{(2)}(p', W) \\ &\quad + ie^2 \int d^4k (\gamma_\nu \frac{1}{\not{p} - \not{k} - W} \gamma_\lambda - \gamma_\nu \frac{1}{\not{p}' - \not{k} - W} \gamma_\lambda) D^{\nu\lambda}(k)] \frac{1}{\not{p} - W} . \end{aligned} \quad (42)$$

Recalling (38), we obtain

$$\Lambda_\mu(p', p)(p'-p)^\mu \equiv 0 , \quad (43)$$

which proves that our G_μ does satisfy the Green-Takahashi identity.

As is well-known, when the photon momentum $p'-p$ vanishes, the Green-Takahashi identity reduces to the original Ward identity,

$$G_\mu(p, p; 0) = - \frac{\partial S(p)}{\partial p^\mu} = \int dW \frac{\rho(W)}{\not{p} - W} \gamma_\mu \frac{1}{\not{p} - W} . \quad (44)$$

Because the very right-hand-side of equation (44) is nothing else but the ansatz (2) in the $p'=p$ limit, our $\Lambda_\mu(p', p)$ should identically vanish in this case. To prove this, consider (39). As $p' \rightarrow p$, the first term is regular. To analyse the second part, let $p' = p + \delta p$, with δp an arbitrary infinitesimal four-vector, then we have

$$\begin{aligned}
& \frac{\not{p}'\gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \Sigma^{(2)}(p, W) - \Sigma^{(2)}(p', W) \frac{\not{p}'\gamma_\mu + \gamma_\mu \not{p}}{p'^2 - p^2} \\
&= \frac{\delta \not{p} \gamma_\mu}{2p \cdot \delta p} \Sigma^{(2)}(p, W) - \Sigma^{(2)}(p, W) \frac{\delta \not{p} \gamma_\mu}{2p \cdot \delta p} - \frac{p_\mu}{p \cdot \delta p} \delta p^\nu \frac{\partial \Sigma^{(2)}(p, W)}{\partial p^\nu} + O(\delta p)
\end{aligned} \tag{45}$$

Recalling (40) and working out the algebras, we can simplify the right-hand-side of (45) to

$$- \gamma_\mu A(p^2, W^2) - 2p_\mu \left[\frac{\partial A(p^2, W^2)}{\partial p^2} \not{p} + \frac{\partial B(p^2, W^2)}{\partial p^2} W \right] + O(\delta p) = - \frac{\partial \Sigma^{(2)}(p, W)}{\partial p^\mu} + O(\delta p) . \tag{46}$$

Therefore, in the limit $\delta p \rightarrow 0$, (39) reduces to

$$\begin{aligned}
\Lambda_\mu(p, p) &= \lim_{\delta p \rightarrow 0} \Lambda_\mu(p + \delta p, p) \\
&= \int dW \rho(W) \left[\frac{1}{\not{p} - W} \right]^2 \left\{ ie^2 \int d^4 k \gamma_\nu \frac{1}{\not{p} - k - W} \gamma_\mu \frac{1}{\not{p} - k - W} \gamma_\lambda D^{\lambda\nu}(k) - \frac{\partial \Sigma^{(2)}(p, W)}{\partial p^\mu} \right\} .
\end{aligned} \tag{47}$$

Finally, by using

$$\frac{\partial}{\partial q^\nu} \frac{1}{\not{q} - m} = - \frac{1}{\not{q} - m} \gamma_\nu \frac{1}{\not{q} - m}$$

it is trivial to prove that

$$ie^2 \int d^4 k \gamma_\nu \frac{1}{\not{p} - k - W} \gamma_\mu \frac{1}{\not{p} - k - W} \gamma_\lambda D^{\nu\lambda}(k) = \frac{\partial \Sigma^{(2)}(p, W)}{\partial p^\mu} \tag{48}$$

Inserting (48) into (47) results in

$$\Lambda_\mu(p, p) \equiv 0 . \tag{49}$$

This confirms the dominance of the lowest order ansatz in the infrared, and explains why the gauge technique is so successful in reproducing the infrared behaviours of the charged particle propagator.

To summarize: we have obtained a spectral representation of the three-point amplitude, which is exact to e^2 order and possesses all the desired properties of the true vertex. The method we have used has the advantage that no ad hoc assumptions are necessary, it falls naturally within the framework of the Dyson-Schwinger equations. By contrast to King's [53] method, the vanishing of the e^2 order correction in the zero photon momentum limit comes out automatically and does not need to be invoked as a restriction.

One more thing: if we represent the three-point Green function by a weighted sum

$$G_\mu(p', p; p'-p) = \int dW \rho(W) g_\mu(p', p|W), \quad (50)$$

the g_μ (even though a closed form for it may not exist) can at least be written as a power series in the coupling constant e^2 ,

$$g_\mu(p', p|W) = g_\mu^{(0)}(p', p|W) + e^2 g_\mu^{(1)}(p', p|W) + e^4 g_\mu^{(2)}(p', p|W) + \dots \quad (51)$$

To this extent we can say both the Delbourgo-West ansatz and our $\Lambda_\mu(p', p)$ are unique.

III.3 THE REFINED GAUGE TECHNIQUE EQUATION AND ITS SOLUTION

In the last section we obtained a spectral representation for the three-point amplitude G_μ , which contains a transverse part and is exact to the e^2 order. In this section we will apply this G_μ to truncate the Dyson-Schwinger equation and solve it for the spectral function.

The Refined Gauge Technique Equation

Inserting (30) and (39) into (4) we once more arrive at the standard gauge technique equation

$$\int dW \frac{\rho(W)}{p-W} [W - m_0 + \Sigma(p, W)] = 0 \quad , \quad (52)$$

by recalling the sum rules (II-95) to get rid of the renormalization constant Z_ψ^{-1} . In (52) the $\Sigma(p, W)$ consists of two parts:

$$\Sigma(p, W) = \Sigma^{(2)}(p, W) + \Sigma_\Lambda(p, W) \quad . \quad (53)$$

The first part $\Sigma^{(2)}(p, W)$ is defined in (38). The $\Sigma_\Lambda(p, W)$ arises from the transverse vertex $\Lambda_\mu(p, p-k)$, it can be decomposed into two parts,

$$\Sigma_\Lambda(p, W) = L(p, W) + I(p, W) \quad , \quad (54)$$

with

$$L(p, W) = e^4 \int d^4 k d^4 \ell \gamma_\nu \frac{1}{p-k-W} \gamma_\mu \frac{1}{p-\ell-k-W} \gamma_\lambda \frac{1}{p-\ell-W} \gamma_\rho D^{\nu\lambda}(k) D^{\mu\rho}(\ell) \quad , \quad (55)$$

$$I(p, W) = ie^2 \int d^4 \ell \left\{ \Sigma^{(2)}(p, W) \frac{\not{p}\gamma_\mu + \gamma_\mu(\not{p}-\ell)}{p^2 - (p-\ell)^2} - \frac{\not{p}\gamma_\mu + \gamma_\mu(\not{p}-\ell)}{p^2 - (p-\ell)^2} \Sigma^{(2)}(p-\ell, W) \right\} \frac{1}{p-\ell-W} \gamma_\rho D^{\mu\rho}(\ell) \quad . \quad (56)$$

In the lowest order gauge technique, we neglected the photon dressing by replacing the full photon propagator with the bare one. But now we should take into account the contributions of the photon vacuum polarization, so that we can make our equation (52) exact to e^4 order. As well-known, the renormalized photon propagator is of the form

$$D_{\mu\nu}(k) = \frac{d(k^2)}{k^2} (-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}) - a \frac{k_\mu k_\nu}{k^4} \quad , \quad (57)$$

with $d(k^2)$ a scalar function of k^2 only. In the small k^2 limit, the full propagator coincides with the bare one,

$$\text{i.e.} \quad d(0) = 1 \quad . \quad (58)$$

Therefore we can express $d(k^2)$ by another function $K(k^2)$ defined by

$$d(k^2) = 1 + 4\left(\frac{e}{4\pi}\right)^2 K(k^2) \quad , \quad (59)$$

with K vanishing while k^2 approaches zero. Because $K(k^2)$ relates to the spectral function itself through the vacuum polarization tensor in (II-44), the inclusion of photon dressing would introduce non-linearity into (52). However, noting that we only require our equation to be exact to e^4 order, a $K(k^2)$ function which agrees with the lowest order perturbation theory is sufficient for our use. We can easily work out $K(k^2)$ by calculating Feynman diagrams; it assumes the following expression

$$K(k^2) = -\frac{k^2}{3} \int_{4m^2}^{\infty} dM^2 \frac{(1+2m^2/M^2) \cdot (1-4m^2/M^2)^{1/2}}{M^2(M^2-k^2)} \quad (60)$$

For later use, we work out the derivative of (60) in the vanishing k^2 limit.

$$\left. \frac{dK(k^2)}{dk^2} \right|_{k^2=0} = -\frac{1}{3} \int_{4m^2}^{\infty} dM^2 \frac{1}{M^4} \left(1 + \frac{2m^2}{M^2}\right) \left(1 - \frac{4m^2}{M^2}\right)^{1/2} = -\frac{1}{15m^2} \quad (61)$$

Now we can write $\Sigma^{(2)}(p, W)$ as

$$\Sigma^{(2)}(p, W) = \Sigma^{(0)}(p, W) + \Sigma_{\gamma}(p, W) \quad (62)$$

$\Sigma^{(0)}(p, W)$ is the lowest order mass correction to an electron of mass W , given by (7) and Σ_{γ} the vacuum polarization contribution, given by

$$\Sigma_{\gamma}(p, W) = -i4e^2 \left(\frac{e}{4\pi}\right)^2 \cdot \int d^4k \gamma_{\mu} \frac{1}{p-k-W} \gamma_{\nu} \frac{K(k^2)}{k^2} (-g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{k^2}) \quad (63)$$

The photon dressing can be discarded in both (55) and (56) because it only contributes higher than order e^4 terms. Thus $I(p, W)$ and $L(p, W)$ reduce to

$$I(p, W) = ie^2 \int d^4k \{ \Sigma^{(0)}(p, W) \frac{\not{p}\gamma_{\mu} + \gamma_{\mu}(\not{p}-\not{k})}{p^2 - (p-k)^2} - \frac{\not{p}\gamma_{\mu} + \gamma_{\mu}(\not{p}-\not{k})}{p^2 - (p-k)^2} \Sigma^{(0)}(p-k, W) \} \frac{1}{p-k-W} \gamma_{\rho} D^{(0)\mu\rho}(k) ,$$

(64)

$$L(p, W) = e^4 \int \bar{d}^4 k \bar{d}^4 \ell \gamma_\mu \frac{1}{\not{p} - \not{\ell} - W} \gamma_\nu \frac{1}{\not{p} - \not{\ell} - \not{k} - W} \gamma_\lambda \frac{1}{\not{p} - \not{k} - W} \gamma_\rho D^{(0)\mu\lambda}(\ell) D^{(0)\nu\rho}(k) .$$

The evaluation of $\Sigma(p, W)$ is obviously very involved. We have carried it out in Appendix A using certain mathematical formulae. The absorptive part of Σ is worked out under a cut-off regularization [70]. It reads

$$\text{Im}\Sigma(p, W) = \text{Im}\Sigma_f(p, W) - 3\left(\frac{e}{4\pi}\right)^2 \text{Im}\Sigma^{(0)}(p, W) \cdot \left[\ln \frac{\Lambda^2}{W^2} + 1\right] , \quad (65)$$

where $\text{Im}\Sigma_f(p, W)$ is finite and given by (A-44); the second term of (65) is ultraviolet cut-off dependent.

To consider equation (52) we first replace \not{p} by a scalar W then take its imaginary part. After inserting (65) into the resultant equation we arrive at

$$\begin{aligned} \pi\epsilon(W)\rho(W)[W - m_0 + \text{Re}\Sigma(W, W)] &= \int dW' \frac{\rho(W')}{W - W'} \{ \text{Im}\Sigma_f(W, W') \\ &- 3\left(\frac{e}{4\pi}\right)^2 \text{Im}\Sigma^{(0)}(W, W') \cdot \left[\ln \frac{\Lambda^2}{W'^2} + 1\right] \} . \end{aligned} \quad (66)$$

Renormalization of the Equation

There exist Λ^2 -dependent terms on both sides of (66), therefore it does need renormalization. To do this, we refer to perturbation theory. Due to the fact that (66) is exact only to order e^4 , we can only expect renormalization to work to this order. Any other higher order divergences will be discarded insofar as they must cancel against higher order transverse corrections. We consider the left-hand-side of (66) first. Realizing that

$$m_0 = m + \delta m , \quad \delta m = \text{Re}\Sigma(m, m) , \quad (67)$$

the terms in the square brackets can be written as

$$W - m + \text{Re}\Sigma^{(0)}(W, W) - \text{Re}\Sigma^{(0)}(m, m) + \text{Re}\Sigma^{(1)}(W, W) - \text{Re}\Sigma^{(1)}(m, m) + O(e^6) , \quad (68)$$

where $\text{Re}\Sigma^{(1)}$ is of e^4 order.

$\text{Re}\Sigma^{(0)}(m,m)$ has been worked out in Appendix A, and yields

$$\begin{aligned} \text{Re}\Sigma^{(0)}(W,W) - \text{Re}\Sigma^{(0)}(m,m) &= \left(\frac{e}{4\pi}\right)^2 \cdot \left\{ -3(W-m) \cdot \left[\ln \frac{\Lambda^2}{m^2} + 1\right] \right. \\ &\quad \left. + (W-m) \left[\frac{3W}{W-m} \ln \frac{W^2}{m^2} + 1\right] \right\}, \end{aligned} \quad (69)$$

therefore the left-hand-side of (66) reads

$$\begin{aligned} &\pi\epsilon(W)\rho(W)(W-m) \left[1 + \left(\frac{e}{4\pi}\right)^2 \cdot \left(\frac{3W}{W-m} \ln \frac{W^2}{m^2} + 1\right)\right] \\ &+ \pi\epsilon(W)\rho(W) \left\{ -3(W-m) \cdot \left(\frac{e}{4\pi}\right)^2 \cdot \left[\ln \frac{\Lambda^2}{m^2} + 1\right] + \text{Re}\Sigma^{(1)}(W,W) - \text{Re}\Sigma^{(1)}(m,m) + O(e^6) \right\}. \end{aligned} \quad (70)$$

Expanding $\rho(W)$ into a perturbative series, we obtain

$$\rho(W) = \delta(W-m) + e^2 \rho_1(W) + e^4 \rho_2(W) + \dots \quad (71)$$

$$\text{where} \quad e^2 \rho_1(W) = \frac{\epsilon(W) \text{Im}\Sigma^{(0)}(W,m)}{\pi(W-m)^2}. \quad (72)$$

Substituting (71) and (72) in (70) and realizing that the term $\text{Re}\Sigma^{(1)}(W,W) - \text{Re}\Sigma^{(1)}(m,m)$ is cancelled by the $\delta(W-m)$, we arrive at

$$\begin{aligned} &\pi\epsilon(W)\rho(W)[W-m_0 + \text{Re}\Sigma(W,W)] \\ &= \pi\epsilon(W)\rho(W)(W-m) \left[1 + \left(\frac{e}{4\pi}\right)^2 \left(\frac{3W}{W-m} \ln \frac{W^2}{m^2} + 1\right)\right] \\ &\quad - 3\left(\frac{e}{4\pi}\right)^2 \cdot \frac{\text{Im}\Sigma^{(0)}(W,m)}{W-m} \left[\ln \frac{\Lambda^2}{m^2} + 1\right] + O(e^6). \end{aligned} \quad (73)$$

To isolate the divergences on the right-hand-side of (66), we apply the expansion (71) to the $\rho(W)$ attached with the cut-off dependent term, and end up with

$$\begin{aligned} &\int dW' \frac{\rho(W')}{W-W'} \left\{ \text{Im}\Sigma_f(W,W') - 3\left(\frac{e}{4\pi}\right)^2 \cdot \text{Im}\Sigma^{(0)}(W,W') \left[\ln \frac{\Lambda^2}{W'^2} + 1\right] \right\} \\ &= \int dW' \frac{\rho(W')}{W-W'} \text{Im}\Sigma_f(W,W') - 3\left(\frac{e}{4\pi}\right)^2 \frac{\text{Im}\Sigma^{(0)}(W,m)}{W-m} \left[\ln \frac{\Lambda^2}{m^2} + 1\right] + O(e^6). \end{aligned} \quad (74)$$

Combining (73) with (74) and leaving out the $O(e^6)$ terms results in the integral equation

$$\pi\epsilon(W)\rho(W)(W-m) \left[1 + \left(\frac{e}{4\pi}\right)^2 \left(\frac{3W}{W-m} \ln \frac{W^2}{m^2} + 1\right)\right] = \int dW' \frac{\rho(W')}{W-W'} \text{Im}\Sigma_f(W, W') . \quad (75)$$

This is exact to e^4 order, finite and linear in $\rho(W)$.

(One may argue that we could have only cancelled the $\ln \frac{\Lambda^2}{m^2}$ term away, leaving the number 1 in the equation. This of course could be one choice, but we have tried to renormalize (66) in the spirit of the so-called modified minimal subtraction scheme [81].)

For the sake of convenience, we insert (A-44) into (75) and spell out the explicit equation

$$\begin{aligned} \epsilon(W)\rho(W)(W-m) \left[1 + \eta \left(\frac{3W}{W-m} \ln \frac{W^2}{m^2} + 1\right)\right] &= \int dW' \frac{\rho(W')}{W-W'} \left\{ \eta \frac{W^2 - W'^2}{W^2} \left[-(a+3)W' + a \frac{W^2 + W'^2}{W} \right] \right. \\ &+ \eta^2 \frac{W^2 - W'^2}{W^2} \left[-(a+3)W' + a \frac{W^2 + W'^2}{W} \right] \frac{W+W'}{W^2} \left[2(4W'-W) \frac{W^2}{W^2 - W'^2} \ln \frac{W^2}{W'^2} \right. \\ &\quad \left. + 2(4W' - \frac{W^2 + W'^2}{W}) \ln \frac{W'^2}{W^2 - W'^2} + W \right] \\ &- \eta^2 \frac{W^2 - W'^2}{W^2} \left[-(a+3) \left(\frac{W'^2}{W} + \frac{4W'^3}{W^2 - W'^2} \ln \frac{W^2}{W'^2} - 3W' \right) \right. \\ &\quad \left. + a \left(\left(\frac{W^2 - W'^2}{W^2} + \frac{4W^2}{W^2 - W'^2} \ln \frac{W^2}{W'^2} \right) W' \right. \right. \\ &\quad \left. \left. + \frac{9W'^2 - 7W^2}{2W} + \frac{4W'^4 - W^4}{W(W^2 - W'^2)} \ln \frac{W^2}{W'^2} \right) \right] \\ &- \eta^2 \frac{W^2 - W'^2}{W^2} \left(-4W' + \frac{W^2 + W'^2}{W} \right) - \frac{2}{15} \eta^2 \frac{(W^2 - W'^2)^3}{W^3 m^2} \\ &- \eta^2 (1-a) \frac{W^2 - W'^2}{W^2} \left[W' \left(\frac{3W^2 - 2W'^2}{W^2} + 2 \frac{W'^4 + W^2 W'^2 - 4W^4}{W^4} \ln \frac{W'^2}{W^2 - W'^2} - \frac{3W'^2}{W^2 - W'^2} \ln \frac{W^2}{W'^2} \right) \right. \\ &\quad \left. + \frac{3W'^2 - W^2}{2W} + 2 \frac{W^2 W'^2 + W^4 - 4W'^4}{W^3} \ln \frac{W'^2}{W^2 - W'^2} \right] \end{aligned}$$

$$\begin{aligned}
& - \eta^2 \frac{W^2 - W'^2}{W^2} \left[8W' \left(2 + \frac{W^2 + W'^2}{2(W^2 - W'^2)} f\left(\frac{W^2}{W^2 - W'^2}\right) + \frac{3(W^2 - W'^2)}{2W^2} \ln \frac{W'^2}{W^2 - W'^2} \right. \right. \\
& \quad \left. \left. - \frac{W^2 + W'^2}{2(W^2 - W'^2)} \ln \frac{W^2}{W'^2} \ln \frac{W'^2}{W^2 - W'^2} \right) \right. \\
& + 2W \left(-2 + \frac{4W'^2}{W^2 - W'^2} f\left(\frac{W^2}{W^2 - W'^2}\right) - 4 \frac{W'^2}{W^2 - W'^2} \ln \frac{W^2}{W'^2} \ln \frac{W'^2}{W^2 - W'^2} \right. \\
& \quad \left. \left. - \frac{4W'^2(W^2 - W'^2)}{W^4} \ln \frac{W'^2}{W^2 - W'^2} \right) \right] \\
& + 4\eta^2 \left[\left(\frac{W^2 + W'^2}{W^2} W' + \frac{2W'^4}{W^3} \right) Z_1(W^2, W'^2) + \left(W' - \frac{W^2 - W'^2}{W} \right) Z_2(W^2, W'^2) \right] \\
& + 8\eta^2 [W' Y_1(W^2, W'^2) + W Y_2(W^2, W'^2)] + 4\eta^2 \frac{W^2 - W'^2}{W^2} \left[-(a+3)W' + a \frac{W^2 + W'^2}{W} \right] \ln \frac{W^2}{W'^2} \} .
\end{aligned}
\tag{75'}$$

Infrared and Ultra-violet Solutions

Although the equation is extremely difficult to solve, it becomes amenable when one goes to the asymptopia. In the infrared region, $W \rightarrow m$, the left-hand-side of (75') reduces simply to

$$\pi \rho(W) (W-m) \left[1 + 7 \left(\frac{e}{4\pi} \right)^2 \right] , \tag{76}$$

but the right-hand-side needs careful considerations. In this region, we have the relationship

$$\left(\int_{-|W|}^{-m} + \int_{+m}^{+|W|} \right) \frac{dW' \rho(W')}{W - W'} \operatorname{Im} \Sigma_f(W, W') \simeq \left(\lim_{\substack{W' \rightarrow m \\ W \rightarrow m+0^+}} \frac{\operatorname{Im} \Sigma_f(W, W')}{W - W'} \right) \cdot \int_m^W dW' \rho(W') ,
\tag{77}$$

as long as the limit is finite. To evaluate this limit, we first note that, the cut of the Y 's starts from $W^2 = 9W'^2$, therefore they do not contribute to the limit. The other terms requiring special considerations are the Z 's. We quote the definitions here from the appendix,

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \int_0^{\epsilon^2} (|W| - |W'|)^2 dq^2 \begin{bmatrix} \frac{1}{W^2 - W'^2 - q^2} \\ \frac{1}{W^2} \end{bmatrix} \times \ln \left| \frac{W^2 - W'^2 - q^2 + [(W^2 - W'^2 + q^2)^2 - 4W^2 q^2]^{\frac{1}{2}}}{W^2 - W'^2 - q^2 - [(W^2 - W'^2 + q^2)^2 - 4W^2 q^2]^{\frac{1}{2}}} \right|. \quad (78)$$

Let $W' = m$, $W = \epsilon + m$, with ϵ an infinitesimal quantity, equation (78) reduces to

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2m\epsilon} \\ \frac{1}{m^2} \end{bmatrix} \int_0^{\epsilon^2} dq^2 \ln \left| \frac{\epsilon + \sqrt{\epsilon^2 - q^2}}{\epsilon - \sqrt{\epsilon^2 - q^2}} \right|.$$

By changing the integration variable $q^2 = \epsilon^2 \xi$ and working out the integrals explicitly, we obtain the leading terms for the Z 's,

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \frac{\epsilon}{2m} \\ \frac{\epsilon^2}{m^2} \end{bmatrix} \cdot \int_0^1 d\xi \ln \frac{1 + \sqrt{1 - \xi}}{1 - \sqrt{1 - \xi}} = \begin{bmatrix} \frac{\epsilon}{m} \\ \frac{2\epsilon^2}{m^2} \end{bmatrix} = \begin{bmatrix} (W - W')/m \\ 2(W - W')^2/m^2 \end{bmatrix}.$$

This result is unique - it does not depend on the way we evaluate it.

With the above considerations and noting that the divergences arising from the terms with $\ln \frac{W'^2}{W^2 - W'^2}$ factor cancel out intrinsically, we obtain

$$\lim_{\substack{W' \rightarrow m \\ W \rightarrow m+0^+}} \frac{\text{Im} \Sigma_f(W, W')}{W - W'} = 2\pi \left(\frac{e}{4\pi}\right)^2 \cdot (a-3) \left[1 + 7\left(\frac{e}{4\pi}\right)^2\right]. \quad (79)$$

Substituting (76) and (79) into (75') we arrive at the familiar equation

$$\rho(W)(W-m) \simeq 2\left(\frac{e}{4\pi}\right)^2 \cdot (a-3) \int_m^W dW' \rho(W')$$

with the solution

$$\rho(W) = R(e^2) \left(\frac{1}{W-m}\right)^{1 - e^2(a-3)/8\pi^2}$$

This exactly agrees with (19).

We can carry out similar analysis in the ultra-violet domain, $W/m \rightarrow \infty$. In the small coupling limit, the following dominant behaviour for $\rho(W)$ can be extracted from the lowest order gauge approximation

$$\rho(W) \sim \epsilon(W) [W(W^2)^{-1+ae^2/16\pi^2} (1+x \ln \frac{W^2}{m^2}) + m.(W^2)^{-1-3e^2/16\pi^2} (1+y \ln \frac{W^2}{m^2})] . \quad (81)$$

Self-consistency of equation (75') in this region leads us to

$$x = y = - \frac{3e^2}{16\pi^2}$$

and this in turn provides the expression for the propagator $S(p)$

$$S(p) \approx [p(p^2)^{-1+ae^2/16\pi^2} (1 - \frac{3e^2}{16\pi^2} \ln \frac{p^2}{m^2}) + m(p^2)^{-1-3e^2/16\pi^2} (1 - \frac{3e^2}{16\pi^2} \ln \frac{p^2}{m^2})] .$$

We should point out that higher order gauge approximations certainly will bring into (81) powers of $\frac{e^2}{16\pi^2} \ln \frac{W^2}{m^2}$. It is hard to see what behaviour we will get after summing up all these powers.

III.4 VACUUM POLARIZATION

In this final section, we want to discuss some aspects of the vacuum polarization utilizing the approximate G_μ given by (30) and (37). The problem of vacuum polarization can be traced back to Dirac's hole theory [67]. A positive energy electron electrostatically repels the electron in the negative energy sea and thereby polarizes the vacuum in its vicinity; the charge density must now be modified by the polarized vacuum, and therefore the interaction nature of two electrons deviates from the Coulomb law at short distances. Hence the effect of vacuum polarization is detectable; in fact, it has been observed in both

hydrogen-like atom spectra and Lamb shift [67].

Any physically observable quantity, such as the vacuum polarization tensor $\pi_{\mu\nu}(k)$, should be gauge independent. The identity (II-71) shows that the longitudinal piece of $D_{\mu\nu}(k)$ is unchanged after radiative corrections are taken into account, where

$$D_{\mu\nu}^{-1}(k) = Z_A(-g_{\mu\nu}k^2 + k_\mu k_\nu) - \frac{1}{a} k_\mu k_\nu + \pi_{\mu\nu}(k) . \quad (82)$$

By transversality we can express $\pi_{\mu\nu}(k)$ as

$$\pi_{\mu\nu}(k) = (-g_{\mu\nu}k^2 + k_\mu k_\nu)\pi(k^2) , \quad (83)$$

without losing generality, and convert (82) into

$$D_{\mu\nu}(k) = \frac{1}{k^2} (-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2})(Z_A + \pi(k^2))^{-1} - a \frac{k_\mu k_\nu}{k^4} . \quad (84)$$

The longitudinal part of $D_{\mu\nu}(k)$ is physically irrelevant, but the transverse piece can be interpreted as the Fourier transform of interacting potential of two unit point charges. As the separation between the charges goes to infinity, one should obtain Coulomb law; that is, in infrared region $k^2 \rightarrow 0$, there is a basic relationship between Z_A and $\pi(0)$

$$Z_A + \pi(0) = 1 . \quad (85)$$

This leads us to a more convenient way of expressing $D_{\mu\nu}(k)$:

$$D_{\mu\nu}(k) = \frac{d(k^2)}{k^2} (-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}) - a \frac{k_\mu k_\nu}{k^4} , \quad (86)$$

with

$$d(k^2) = [1 + (\pi(k^2) - \pi(0))]^{-1} . \quad (87)$$

(87) should be both gauge invariant and ultra-violet cut-off independent.

However, the $d(k^2)$ obtained by lowest order gauge technique meets these requirements only to the e^2 order. As can be easily seen, it

depends on the gauge parameter "a" up to e^4 order through both the renormalization constant Z_ψ and the spectral function. Hopefully the refined gauge technique with the inclusion of transverse vertices can overcome this criticism. We analyse this problem below.

After the $G_\mu(p', p; p'-p)$ given in (30) is used, the vacuum polarization tensor takes the form

$$\pi_{\mu\nu}(k) = iZ_\psi e^2 \int dW \rho(W) \int \bar{d}^4 p \operatorname{tr} \left\{ \frac{1}{\not{p} + \not{k} - W} \gamma_\mu \frac{1}{\not{p} - W} \gamma_\nu + \lambda_\mu(p+k, p|W) \gamma_\nu \right\}; \quad (88)$$

where $\lambda_\mu(p+k, p|W)$ is defined by

$$\Lambda_\mu(p+k, p|W) = \int dW \rho(W) \lambda_\mu(p+k, p|W). \quad (89)$$

Because the charge renormalization constant is gauge invariant, the derivative of (88) with respect to the gauge parameter a should identically vanish, at least to the order to which it is exact, that is, e^4 . Noting that, the gauge parameter 'a' is always attached to the coupling constant, we have the very important property of Z_ψ and $\rho(W)$ that their derivatives with respect to 'a' are at least of e^2 order. Therefore to $O(e^4)$,

$$\begin{aligned} \frac{\partial \pi(k^2)}{\partial a} &= e^2 \frac{\partial Z_\psi^{(1)}}{\partial a} \pi^{(1)}(k^2|m^2) + \int dW \frac{\partial e^2 \rho_1(W)}{\partial a} \pi^{(1)}(k^2|W^2) \\ &\quad - \frac{ie^2}{3k^2} \int \bar{d}^4 p \operatorname{tr} \left[\frac{\partial}{\partial a} \lambda_\mu(p+k, p|m) \gamma^\mu \right]. \end{aligned} \quad (90)$$

Above, $e^2 Z_\psi^{(1)}$ and $e^2 \rho_1(W)$ denote the first order corrections of Z_ψ and ρ respectively, and $\pi^{(1)}(k^2|W^2)$ is defined by

$$\pi_{\mu\nu}^{(1)}(k|W) = ie^2 \int \bar{d}^4 k \operatorname{tr} \left(\frac{1}{\not{p} + \not{k} - W} \gamma_\mu \frac{1}{\not{p} - W} \gamma_\nu \right) = (-g_{\mu\nu} k^2 + k_\mu k_\nu) \pi^{(1)}(k^2|W^2). \quad (91)$$

We consider $Z_\psi^{(1)}$ first. The well-known sum rule

$$Z_\psi^{-1} = \int dW \rho(W)$$

gives the perturbative result,

$$e^2 Z_\psi^{(1)} = - \int dW e^2 \rho_1(W) . \quad (92)$$

This integral is both infrared and ultra-violet divergent, hence we need an unambiguous way to express it. The relationship (72) comes to our aid. More careful analysis leads us to the following equation

$$e^2 Z_\psi^{(1)} = - \int dW e^2 \rho_1(W) = - \frac{1}{\pi} \frac{\partial}{\partial M} \text{Re} \Sigma^{(0)}(M, m) |_{M=m} . \quad (93)$$

(93) enables us to evaluate $\frac{\partial}{\partial a} e^2 Z_\psi^{(1)}$ through the original definition of $\Sigma^{(0)}(p, m)$. It is not too difficult to arrive at

$$e^2 \frac{\partial Z_\psi^{(1)}}{\partial a} = i e^2 \int d^4 k \frac{1}{k} , \quad (94)$$

from (93). (Needless to say, such a badly divergent integral can only have formal meaning.)

The second term of (90) is simpler to evaluate. Noting that only the even part of $e^2 \rho_1(W)$ contributes to it, we obtain, after some elementary manipulations,

$$\int dW \frac{\partial}{\partial a} e^2 \rho_1(W) \pi^{(1)}(k^2 | W^2) = \left(\frac{e}{4\pi}\right)^2 \cdot \int_{m^2}^{\infty} dW^2 \left[\frac{W^2 + m^2}{W^2(W^2 - m^2)} - \frac{m^2}{W^4} \right] \pi^{(1)}(k^2 | W^2) . \quad (95)$$

Because of the complexity of $\lambda_\mu(p+k, p|W)$ and the overlapping divergences, the third term of (90), which we will denote by $E(k^2)$, is extremely difficult to evaluate. Fortunately we do not have to calculate it explicitly, an implicit result will be sufficient for our use as long as it enables us to compare $E(k^2)$ with the other two terms.

From the definitions (37) and (89), we can easily extract out an explicit expression for $\lambda_\mu(p+k, p|W)$. Then we differentiate it with respect to the gauge parameter a and repeatedly use the formula

$$\frac{1}{d-M} \not{x} \frac{1}{d-\not{x}-M} = \frac{1}{d-\not{x}-M} - \frac{1}{d-M} ;$$

this results in

$$\begin{aligned} \frac{\partial \lambda_\mu(p', p|m)}{\partial a} = & -ie^2 \int \frac{d^4 t}{t^4} \left\{ \frac{1}{p' - t - m} \gamma_\mu \frac{1}{p - t - m} \right. \\ & \left. + \left[\frac{1}{p' - t - m} \frac{p' \gamma_\mu + \gamma_\mu p}{p'^2 - p^2} - \frac{p' \gamma_\mu + \gamma_\mu p}{p'^2 - p^2} \frac{1}{p - t - m} \right] \right\}. \quad (96) \end{aligned}$$

Applying (96) to the third term of (90) and adopting dimensional regularization, so that translation of variables will be allowed, we arrive at

$$\begin{aligned} -3k^2 E(k^2) = & -ie^2 \int \frac{d^{2\ell} t}{t^4} \pi_\mu^{(0)}(k|m) \\ & - ie^2 \int \frac{d^{2\ell} p}{p^2} \text{tr}\{\gamma^\mu (ie^2) \int \frac{d^{2\ell} t}{t^4} \left[\frac{p' - t}{(p' - t)^2 - m^2} \cdot \frac{p' \gamma_\mu + \gamma_\mu p}{p'^2 - p^2} \right. \right. \\ & \left. \left. - \frac{p' \gamma_\mu + \gamma_\mu p}{p'^2 - p^2} \cdot \frac{p - t}{(p - t)^2 - m^2} \right] \right\}, \quad (97) \end{aligned}$$

with $p' = p + k$, and $\ell \rightarrow 2$ being understood. Defining the following scalar function

$$\sigma(p^2, m^2) = ie^2 \int \frac{d^{2\ell} t}{t^4} \cdot \frac{p - t}{p[(p - t)^2 - m^2]}, \quad (98)$$

and expressing it via the dispersion relation

$$\sigma(p^2, m^2) = \frac{1}{\pi} \int dW^2 \frac{\text{Im}\sigma(p^2, W^2)}{W^2 - p^2 - i0^+},$$

we can, after some careful algebraic juggling, manipulate (97) to

$$E(k^2) = -ie^2 \int \frac{d^{2\ell} t}{t^4} \pi^{(1)}(k^2|m^2) - \frac{1}{\pi} \int dW^2 \text{Im}\sigma(W^2, m^2) \pi^{(1)}(k^2|W^2). \quad (99)$$

The imaginary part of $\sigma(p^2, m^2)$ can be worked out from (98). It reads

$$\text{Im}\sigma(p^2, m^2) = \pi \left(\frac{e}{4\pi} \right)^2 \left[\frac{p^2 + m^2}{p^2(p^2 - m^2)} - \frac{m^2}{p^4} \right] \theta(p^2 - m^2). \quad (100)$$

By substituting (94), (95) and (99), (100) into (90), a dramatic cancellation happens, which leaves us with

$$\frac{\partial \pi(k^2)}{\partial a} \equiv 0 \quad . \quad (101)$$

This is what we sought to prove.

(101) indicates that the gauge properties of the spectral function $\rho(W)$ are improved by the introduction of transverse vertices. It follows that the gauge covariance is ameliorated at intermediate momenta, although this problem will not be tackled here.

IV. THE GAUGE TECHNIQUE IN SCALAR ELECTRODYNAMICS

The present chapter is devoted to the applications of the gauge technique to scalar electrodynamics. We first give a detailed review of the lowest order gauge technique [83] and Parker's method [54] of introducing transverse vertices, then extend Parker's work to arbitrary covariant gauges. We achieve an unambiguous gauge technique equation, and solve it in asymptopia. Its infrared solution gives the standard infrared singularity of the charged meson propagator. The transverse vertex is also used to study the photon vacuum polarization tensor; it turns out that the vacuum polarization tensor is gauge invariant up to e^4 order. At the end of the chapter we study the lowest order gauge technique in the axial gauge, and obtain an approximate spectral function valid at all momenta.

IV.1 THE LOWEST ORDER GAUGE TECHNIQUE IN COVARIANT GAUGES

Shortly after the appearance of the paper by Delbourgo and West [35] on spinor electrodynamics, Delbourgo applied the methods established in that paper to covariant gauge scalar electrodynamics [83]. We present his work here in detail.

Let us recall the gauge identities which were derived in chapter II,

$$k^\mu G_\mu(p', p; k) = \Delta(p) - \Delta(p') , \quad (1)$$

$$k'^\nu G_{\nu\mu}(p', p; -k', k) = G_\mu(p+k, p; k) - G_\mu(p', p'-k; k) . \quad (2)$$

We 'solve' them by making the following ansatz

$$G_\mu^{(0)}(p', p; k) = \int dW^2 \rho(W^2) \frac{1}{p'^2 - W^2} (p' + p)_\mu \frac{1}{p^2 - W^2} \quad (3)$$

$$G_{\nu\mu}^{(0)}(p', p; -k', k) = \int dW^2 \rho(W^2) \frac{1}{p'^2 - W^2} \left[2g_{\mu\nu} - \frac{(2p' + k')_\nu (2p + k)_\mu}{(p + k)^2 - W^2} - \frac{(2p - k')_\nu (2p' - k)_\mu}{(p - k')^2 - W^2} \right] \frac{1}{p^2 - W^2} . \quad (4)$$

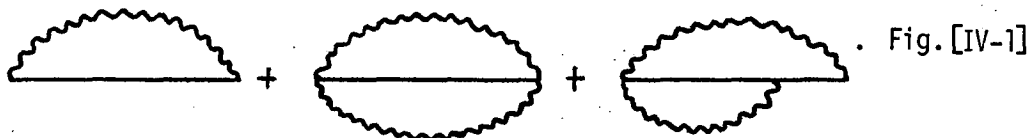
Both (3) and (4) meet the requirement that when we put them on the meson mass shell by setting $\rho(W^2) = \delta(W^2 - m^2)$, they reduce to the corresponding bare quantities. Following the same procedure as in spinor electrodynamics, we apply (4) and (3) to the Dyson-Schwinger equation

$$Z_\phi^{-1} = \Delta(p)(p^2 - m_0^2) - ie^2 \int d^4k G_\mu(p, p-k)(2p-k)_\nu D^{\mu\nu}(k) + 2e^4 \int d^4k d^4k' G_{\mu\nu}(p, p-k-k'; k, k') D^{\mu\lambda}(k) D^{\nu\sigma}(k') g_{\lambda\sigma} + \text{photon tadpole}, \quad (5)$$

and arrive at

$$Z_\phi^{-1} = \int dS \frac{\rho(S)}{p^2 - S} [p^2 - m_0^2 + \Sigma(p^2, S)] + \text{photon tadpole}, \quad (6)$$

with $\Sigma(p^2, S)$ the mass correction to a meson of mass squared S . In diagrammatic form $\Sigma(p^2, S)$ can be expressed as



Here the photon dressing does not arise. After carrying out the necessary renormalizations in (6), and picking out the pole term in $\rho(W^2)$ by decomposing it into

$$\rho(W^2) = \delta(W^2 - m^2) + \sigma(W^2) , \quad (7)$$

the imaginary part of the resultant equation reads

$$(S - m^2)\sigma(S) = \frac{\text{Im}\Sigma^{(0)}(S, m^2)}{\pi(S - m^2)} + \int dS' \sigma(S') \frac{\text{Im}\Sigma^{(0)}(S, S')}{\pi(S - S')} . \quad (8)$$

The absorptive part of $\Sigma(p^2, W^2)$ was worked out explicitly by Delbourgo [83]. However there exists a minor error in the e^4 order part of his

result (which is corrected later), so we merely restate the e^2 order piece here,

$$\text{Im}\Sigma^{(0)}(p^2, W^2) = \pi \left(\frac{e}{4\pi}\right)^2 (a-3) \frac{p^4 - W^4}{p^2} \theta(p^2 - W^2) . \quad (9)$$

In non-Yennie gauges, $a \neq 3$, (9) gives the dominant contribution to $\text{Im}\Sigma$.

Delbourgo applied (9) to (8) and converted the resultant equation to differential form. The solution, which satisfies appropriate boundary conditions and incorporates an infrared cut-off, involves the hypergeometric function F and reads

$$\sigma(W^2) = \frac{2\xi}{W^2 - m^2} \left(\frac{W^2 - m^2}{\mu^2}\right)^{2\xi} F(a+1, b+1; c; 1 - \frac{W^2}{m^2}) \theta(W^2 - m^2) , \quad (10)$$

with

$$\xi = (a-3) \left(\frac{e}{4\pi}\right)^2, \quad a = \frac{1}{2}[-1 + 2\xi - (1 + 4\xi^2)^{\frac{1}{2}}], \quad b = \frac{1}{2}[-1 + 2\xi + (1 + 4\xi^2)^{\frac{1}{2}}], \quad c = 2\xi.$$

By utilizing (10) the meson propagator itself can be determined, if a and b are approximated to the first order in e^2 ,

$$\Delta(p) = \frac{1}{p^2 - m^2} - \frac{1}{m^2} \left(\frac{m^2}{\mu^2}\right)^{2\xi} \Gamma(1-\xi)\Gamma(2-\xi)\Gamma(1+2\xi) F(1-\xi, 2-\xi; 2; \frac{p^2}{m^2}) . \quad (11)$$

As pointed out before, the decomposition (7) is not true in the infrared region in non-Yennie gauges. Realizing this fact the author with West [37] solved the infrared equation in a general covariant gauge without breaking p into a pole and cut. They obtained

$$\Delta(p) \approx (p^2 - m^2 + i0^+)^{-1 + e^2(a-3)/8\pi^2} . \quad (12)$$

Equation (12) has the same power law structure as the infrared spinor particle propagator. The lowest order gauge technique was also applied to vector electrodynamics [38] to explore the infrared behaviour of the charged vector particle propagator and it turned out to obey the same exponential law,

$$\Delta_{\mu\nu}(p) \approx (-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}) \cdot (p^2 - m^2 + i0^+)^{-1+2(a-3)} \cdot (\frac{e}{4\pi})^2. \quad (13)$$

This led to the conjecture that such behaviour was valid for any field regardless of spin. (Detailed discussions of vector quantum electrodynamics are out of our scope and the interested reader is referred to the relevant literature [84] [38].)

IV.2 A REVIEW OF PARKER'S WORK

In scalar electrodynamics, a significant step was made by C.N. Parker for introducing the transverse vertex into the gauge technique [54]. By truncating the three-point Green function Dyson-Schwinger equations to e^2 order he obtained a non-perturbative approximation for G_μ which contained an order e^2 purely transversal part. Confining himself to $a=1$ gauge, he utilized the G_μ to investigate the spectral function at asymptopia.

Starting with the Dyson-Schwinger equation (II-48), Parker discarded the transverse corrections to both G_μ and $G_{\mu\nu}$ because they were known to belong to higher orders; the equation was recast as

$$(p'^2 - m_0^2)G_\mu(p', p) = (p' + p)_\mu \Delta(p) - 2ie^2 \int d^4t g_{\mu\sigma} D_{(0)}^{\sigma\pi}(t) G_\pi^{(0)}(p-t, p) \\ - ie^2 \int d^4t (2p'-t)_\lambda D_{(0)}^{\lambda\sigma}(t) G_{\sigma\mu}^{(0)}(p'-t, p; -t, p'-p) \quad (14)$$

where $G_\pi^{(0)}$ and $G_{\sigma\mu}^{(0)}$ are given by (3) and (4), respectively. Upon making the following replacement $p' \rightarrow -p$, $p \rightarrow -p'$ in (14), a new equation arises

$$(p^2 - m_0^2)G_\mu(-p, -p') = -(p' + p)_\mu \Delta(p') - 2ie^2 \int d^4t g_{\mu\sigma} D_{(0)}^{\sigma\pi}(t) G_\pi^{(0)}(-p'-t, -p') \\ + ie^2 \int d^4t (2p+t)_\lambda D_{(0)}^{\lambda\sigma}(t) G_{\sigma\mu}^{(0)}(-p-t, -p'; -t, p'-p). \quad (14')$$

Parker employed the charge conjugate properties of G_μ and $G_{\mu\nu}$, i.e.

$$G_\mu(p', p; p' - p) = -G_\mu(-p, -p'; p' - p) \quad (15)$$

$$G_{\mu\nu}(p', p; -\ell', \ell) = G_{\mu\nu}(-p, -p'; -\ell', \ell), \quad (16)$$

in (14) and (14'), and arrived at

$$\begin{aligned} (p'^2 - p^2)G_\mu(p', p) &= (p' + p)_\mu (p' - p)^\nu G_\nu^{(0)}(p', p) \\ &+ 2ie^2 \int \bar{d}^4 t g_{\mu\sigma} D_{(0)}^{\sigma\pi}(t) (p' - p + t)^\nu \cdot G_{\pi\nu}^{(0)}(p', p; -t, p' - p + t) \\ &+ ie^2 \int \bar{d}^4 t D_{(0)}^{\lambda\sigma}(t) [(2p + t)_\lambda G_{\sigma\mu}^{(0)}(p', p + t; -t, p' - p) \\ &\quad - (2p' - t)_\lambda G_{\sigma\mu}^{(0)}(p' - t, p; -t, p' - p)] \end{aligned} \quad (17)$$

Defining the longitudinal and transverse part of G_μ by

$$\begin{aligned} G_\mu^L(p', p) &= L_{\mu\nu}(p', p) G^\mu(p', p) \\ G_\mu^T(p', p) &= T_{\mu\nu}(p', p) G^\mu(p', p) \end{aligned} \quad (18)$$

with L and T two pseudo-projection operators given by

$$\begin{aligned} L_{\mu\nu}(p', p) &= (p' - p)_\mu (p' + p)_\nu / (p'^2 - p^2) \\ T_{\mu\nu}(p', p) &= g_{\mu\nu} - L_{\mu\nu}(p', p), \end{aligned} \quad (19)$$

he proved that G_μ^L coincided with the $G_\mu^{(0)}$ given in (3). His transverse amplitude assumes the expression

$$\begin{aligned} G_\mu^T(p', p; p' - p) &= ie^2 T_{\mu\nu}^\nu(p', p) \cdot \int dW^2 \rho(W^2) \frac{1}{(p'^2 - W^2)(p^2 - W^2)} \\ &\times \int \bar{d}^4 t D_{(0)}^{\sigma\pi}(t) \left\{ \frac{(p' + p + 2t)_\nu (2p + t)_\sigma (2p' + t)_\pi}{[(p + t)^2 - W^2][(p' + t)^2 - W^2]} - 2g_{\nu\sigma} \left[\frac{(2p' + t)_\pi}{(p' + t)^2 - W^2} + \frac{(2p + t)_\pi}{(p + t)^2 - W^2} \right] \right\}. \end{aligned} \quad (20)$$

Parker claimed that G_μ^T vanished in the zero photon momentum limit without

proof. The remainder of his work was to use (20) to solve the Dyson-Schwinger equation (5) in the Feynman gauge. As usual the gauge technique equation assumes the form

$$\pi\rho(p^2)[p^2 - m_0^2 + \text{Re}\Sigma(p^2, p^2)] = \int dW^2 \rho(W^2) \frac{\text{Im}\Sigma(p^2, W^2)}{p^2 - W^2}, \quad (21)$$

with $\Sigma(p^2, W^2)$ the mass correction. Employing Parker's convention that the transverse vertices (that is, proper vertices multiplied by $T_{\mu\nu}$) are enclosed in dashed-line rectangles, the $\Sigma(p^2, W^2)$ can be displayed diagrammatically as

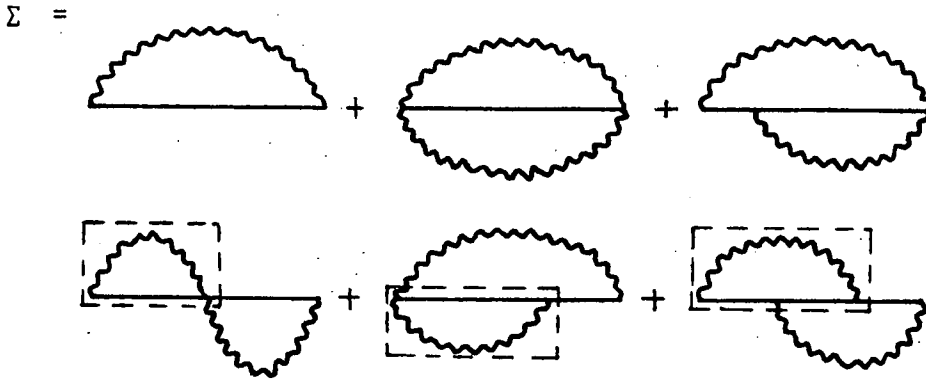


Fig. [IV-2]

In this equation photon vacuum polarization effects are neglected.

Parker explicitly worked out the absorptive part of Σ by applying Cutkosky-Nakanishi [82] cutting rules under dimensional regularization [71, 72, 73]. We quote the result here

$$\begin{aligned} \text{Im}\Sigma(p^2, W^2) = & \pi\eta\theta(p^2 - W^2)[2(W^4 - p^4) + \eta\{6(p^4 - W^4)(\lim_{\ell \rightarrow 2} \frac{1}{(2-\ell)} - \gamma - \ln(\frac{W^2}{4\pi m^2})) \\ & - 2(p^2 - W^2)(p^2 + 7W^2) - 4p^2(p^2 + 3W^2)\ln(\frac{p^2}{W^2}) + 2(p^2 - W^2)\frac{(p^4 + 14p^2W^2 + W^4)}{p^2}\ln(\frac{p^2 - W^2}{W^2}) \\ & - (p^2 + 3W^2)^2(2\ln(\frac{p^2}{W^2})\ln(\frac{p^2 - W^2}{W^2}) + 3f(\frac{p^2}{W^2}))\}]/p^2 + \theta(p^2 - 9W^2)I \end{aligned} \quad (22)$$

where $f(x)$ is the Spence function [85] defined by

$$f(x) = \int_1^x dt \frac{\ln t}{1-t} , \quad (23)$$

and γ is Euler's constant. $I(p^2, W^2)$ is a very complicated function which can only be represented as an integral. For its representation see equation (B-28).

In renormalizing (21), Parker fell back on perturbation theory. He expanded $\rho(W^2)$ in a power series in e^2

$$\rho(W^2) = \delta(W^2 - m^2) + e^2 \rho_1(W^2) + e^4 \rho_2(W^2) + \dots \quad (24)$$

and used (24) as well as the fact that $\delta m^2 = \text{Re} \Sigma(m^2, m^2)$ in (21), to obtain, exact to order e^4 ,

$$\begin{aligned} & \pi \rho(p^2) [p^2 - m_0^2 + \text{Re} \Sigma(p^2, p^2)] \\ & \simeq \pi \rho(p^2) (p^2 - m^2) + \pi e^2 \rho_1(p^2) [\text{Re} \Sigma^{(0)}(p^2, p^2) - \text{Re} \Sigma^{(0)}(m^2, m^2)] , \end{aligned} \quad (25)$$

with $\Sigma^{(0)}(p^2, m^2)$ the lowest order mass correction. Next, substituting the following equation,

$$\text{Re} \Sigma^{(0)}(p^2, p^2) - \text{Re} \Sigma^{(0)}(m^2, m^2) = \eta \cdot (m^2 - p^2) \left\{ 3 \left[\lim_{\ell \rightarrow 2} \frac{1}{2-\ell} - \gamma + \ln(4\pi) \right] + 7 \right\} , \quad (26)$$

into (25), he arrived at

$$\begin{aligned} \pi \rho(p^2) [p^2 - m_0^2 + \text{Re} \Sigma(p^2, p^2)] & \simeq \pi \rho(p^2) (p^2 - m^2) \cdot [1 - 7\eta] \\ & + 6\pi \eta^2 \frac{p^2 + m^2}{p^2} \left[\lim_{\ell \rightarrow 2} \frac{1}{2-\ell} - \gamma + \ln(4\pi) \right] . \end{aligned} \quad (27)$$

The divergences on the right-hand-side of (21) were separated from the finite terms by replacing $\rho(W^2)$ with $\delta(W^2 - m^2)$ in the infinite terms,

$$\begin{aligned} & \int dW^2 \frac{\rho(W^2)}{p^2 - W^2} \left[6\pi \eta^2 \frac{p^4 - W^4}{p^2} \left(C - \ln \frac{W^2}{m^2} \right) + \text{finite terms} \right] \\ & + 6\pi \eta^2 \frac{p^2 + m^2}{p^2} \left[\lim_{\ell \rightarrow 2} \frac{1}{2-\ell} - \gamma + \ln(4\pi) \right] . \end{aligned} \quad (28)$$

The finite constant C was introduced to take account of the ambiguity in the finite part. Combining (27) with (28) and cancelling the divergences led him to the final equation

$$\begin{aligned}
 (p^2 - m^2) \cdot (1 - 7\eta) \rho(p^2) &= \eta \int_{m^2}^{p^2} \frac{dW^2 \rho(W^2)}{2(p^2 - W^2) p^2} (2(W^4 - p^4) + \eta \{6(p^4 - W^4) \cdot \\
 &[C - \ln(\frac{W^2}{m^2})] - 2(p^2 - W^2)(p^2 + 7W^2) - 4p^2(p^2 + 3W^2) \ln(\frac{p^2}{W^2}) \\
 &- (p^2 + 3W^2)^2 (2 \ln(\frac{p^2}{W^2}) \ln(\frac{p^2 - W^2}{W^2}) + 3f(\frac{p^2}{W^2})) \\
 &+ 2(p^2 - W^2) \frac{(p^4 + 14p^2 W^2 + W^4)}{p^2} \ln(\frac{p^2 - W^2}{W^2}) \}) + \theta(p^2 - 9m^2) \int_{m^2}^{p^2/9} \frac{dW^2 \rho(W^2) I}{\pi(p^2 - W^2)}.
 \end{aligned} \quad (29)$$

To determine the arbitrary constant C , he required that equation (29) reproduce the standard infrared behaviour and found that $C=1$ satisfied this requirement.

In the small coupling limit, he solved (29) in the ultra-violet region. Adopting the dimensionless variables

$$x = \frac{p^2}{m^2}, \quad y = \frac{W^2}{m^2}, \quad \phi(x) = m^2 \rho(p^2) \quad (30)$$

he approximated (29) by

$$\begin{aligned}
 (1 - 7\eta)x^2 \phi(x) &= \eta \int_1^x dy \phi(y) (-2(x+y) + \eta \{-6(x+y) \ln y + 4(x-2y) \\
 &+ 2 \frac{(x^2 + 14xy + y^2)}{x} \ln(\frac{x-y}{y}) - \frac{1}{(x-y)} [4x(x+3y) \ln(\frac{x}{y}) \\
 &+ (x+3y)^2 (2 \ln(\frac{x}{y}) \ln(\frac{x-y}{y}) + 3f(\frac{x}{y})) \}]) + \int_1^{x/9} dy \frac{\phi(y) J(x, y) x}{(x-y)}, \\
 m^2 J(x, y) &= I(p^2, W^2).
 \end{aligned} \quad (31)$$

Trying the ansatz

$$\phi(x) \sim x^a (1 + b \ln x), \quad x \rightarrow \infty \quad (32)$$

he claimed a consistent solution was

$$\phi(x) \sim x^{-1-2\eta(1-6\eta^2 \ln x)} , \quad (33)$$

and concluded that: the dominant ultra-violet behaviour was described by the power law obtained from the lowest order gauge technique; the inclusion of transverse vertices only yielded subdominant corrections to this behaviour.

IV.3 SCALAR ELECTRODYNAMICS IN ARBITRARY COVARIANT GAUGES

This section is devoted to the extension of Parker's work to a general covariant gauge characterized by a parameter a . For the sake of convenience, we will slightly change the form of G_μ , so that it will be comparable to the three-point Green function in spinor electrodynamics.

Transverse Amplitude

Abandoning the pseudo-projection operators, we merely divide both sides of (17) by the scalar function $p'^2 - p^2$. The first term ends up as the usual ansatz, and we denote the e^2 order term by $\Lambda_\mu(p', p)$. Thus we obtain

$$G_\mu(p', p) = \int dW^2 \frac{\rho(W^2)}{p'^2 - W^2} (p' + p)_\mu \frac{1}{p^2 - W^2} + \Lambda_\mu(p', p) \quad (34)$$

with Λ_μ given by

$$\begin{aligned} \Lambda_\mu(p', p) = & \int dW^2 \rho(W^2) \frac{1}{p'^2 - W^2} \left\{ ie^2 \int \bar{d}^4 t D_{(0)}^{\sigma\pi}(t) \left[\frac{(p' + p + 2t)_\mu (2p + t)_\sigma (2p' + t)_\pi}{[(p + t)^2 - W^2][(p' + t)^2 - W^2]} \right. \right. \\ & - 2g_{\mu\sigma} \left(\frac{(2p' + t)_\pi}{(p' + t)^2 - W^2} + \frac{(2p + t)_\pi}{(p + t)^2 - W^2} \right) \Big] \\ & + \frac{(p' + p)_\mu}{p'^2 - p^2} ie^2 \int \bar{d}^4 t \left[\frac{(2p' + t)_\sigma (2p' + t)_\pi}{(p' + t)^2 - W^2} \right. \\ & \left. \left. - \frac{(2p + t)_\sigma (2p + t)_\pi}{(p + t)^2 - W^2} \right] D_{(0)}^{\sigma\pi}(t) \right\} \frac{1}{p^2 - W^2} . \quad (35) \end{aligned}$$

By realizing that

$$\Sigma^{(0)}(p^2, w^2) = -ie^2 \int d^4t D_{(0)}^{\sigma\pi}(t) \cdot \frac{(2p+t)_\sigma (2p+t)_\pi}{(p+t)^2 - w^2}, \quad (36)$$

we can rewrite (35) as

$$\begin{aligned} \Lambda_\mu(p', p) = & \int dw^2 \rho(w^2) \frac{1}{p'^2 - w^2} \{ ie^2 \int d^4t D_{(0)}^{\sigma\pi}(t) [\frac{(p'+p+2t)_\mu (2p+t)_\sigma (2p'+t)_\pi}{[(p+t)^2 - w^2][(p'+t)^2 - w^2]} \\ & - 2g_{\mu\sigma} (\frac{(2p'+t)_\pi}{(p'+t)^2 - w^2} + \frac{(2p+t)_\pi}{(p'+t)^2 - w^2})] \\ & - (p'+p)_\mu \frac{\Sigma^{(0)}(p'^2, w^2) - \Sigma^{(0)}(p^2, w^2)}{p'^2 - p^2} \} \frac{1}{p^2 - w^2}, \quad (37) \end{aligned}$$

and it becomes quite obvious that at $p'^2 = p^2$ no fictitious pole exists.

Another important feature of (37) is that it is transversal to the photon momentum $p' - p$, as can be easily shown by multiplying it with $(p' - p)^\mu$.

Therefore our G_μ is equivalent to the one Parker obtained.

As in spinor electrodynamics, the differential Ward identity

$$G_\mu(p, p) = \int dw^2 \rho(w^2) \frac{2p_\mu}{(p^2 - w^2)^2} \quad (38)$$

requires that $\Lambda_\mu(p', p)$ vanish as $p' \rightarrow p$. In fact (37) meets this requirement. An explicit proof of this is given below. Noting

$$\lim_{p' \rightarrow p} \frac{\Sigma^{(0)}(p'^2, w^2) - \Sigma^{(0)}(p^2, w^2)}{p'^2 - p^2} = \frac{\partial \Sigma^{(0)}(p^2, w^2)}{\partial p^2},$$

in the $p' \rightarrow p$ limit, (37) reduces to

$$\begin{aligned} \Lambda_\mu(p, p) = & \int dw^2 \rho(w^2) \frac{1}{p^2 - w^2} \{ ie^2 \int d^4t D_{(0)}^{\sigma\pi}(t) [\frac{2(p+t)_\mu (2p+t)_\sigma (2p+t)_\pi}{[(p+t)^2 - w^2]^2} \\ & - 4g_{\mu\sigma} \frac{(2p+t)_\pi}{(p+t)^2 - w^2}] - 2p_\mu \frac{\partial \Sigma^{(0)}(p^2, w^2)}{\partial p^2} \} \frac{1}{p^2 - w^2}. \quad (39) \end{aligned}$$

It is fairly straight-forward (recalling the definition (36)) to see that

$$\begin{aligned}
 & ie^2 \int \bar{d}t D_{(0)}^{\sigma\pi}(t) \left\{ \frac{2(p+t)_\mu (2p+t)_\sigma (2p+t)_\pi}{[(p+t)^2 - W^2]^2} - 4g_{\mu\sigma} \frac{(2p+t)_\pi}{(p+t)^2 - W^2} \right\} \\
 & = - \frac{\partial}{\partial p^\mu} \left[ie^2 \int \bar{d}t D_{(0)}^{\sigma\pi}(t) \frac{(2p+t)_\sigma (2p+t)_\pi}{(p+t)^2 - W^2} \right] = 2p^\mu \frac{\partial \Sigma^{(0)}(p^2, W^2)}{\partial p^2} . \quad (40)
 \end{aligned}$$

Inserting (40) into (39) we arrive at

$$\lim_{p' \rightarrow p} \Lambda_\mu(p', p) = 0 , \quad (41)$$

that is, $\Lambda_\mu(p', p)$ identically vanishes in the zero photon momentum limit, and (34) satisfies the Ward identity.

Refined Gauge Technique Equation

By going through the standard procedure the integral equation

$$\pi_\rho(p^2) [p^2 - m_0^2 + \text{Re}\Sigma(p^2, p^2)] = \int dW^2 \rho(W^2) \frac{\text{Im}\Sigma(p^2, W^2)}{p^2 - W^2} \quad (42)$$

is once again reproduced, with

$$\begin{aligned}
 \Sigma(p^2, W^2) &= \Sigma^{(0)}(p^2, W^2) - ie^2 \int \bar{d}^4 q \frac{K(q^2)}{q^2} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \frac{(2p-q)^\mu (2p-q)^\nu}{(p-q)^2 - W^2} \\
 &+ e^4 \int \bar{d}^4 t \bar{d}^4 t' g_{\mu\nu} D_{(0)}^{\mu\alpha}(t') D_{(0)}^{\nu\beta}(t) \frac{1}{(p-t-t')^2 - W^2} \\
 &\times \left\{ 2g_{\alpha\beta} - \frac{[2(p-t)-t']_\alpha (2p-t)_\beta}{(p-t)^2 - W^2} - \frac{[2(p-t')-t]_\beta (2p-t')_\alpha}{(p-t')^2 - W^2} \right\} \quad (43) \\
 &+ ie^2 \int \bar{d}^4 q \frac{(2p-q)_\mu D_{(0)}^{\mu\nu}(q)}{(p-q)^2 - W^2} \left\{ -ie^2 \int \bar{d}^4 t \left[\frac{(2p-q+2t)_\nu (2p+t)_\sigma (2p-2q+t)_\pi}{[(p+t)^2 - W^2][(p-q+t)^2 - W^2]} \right. \right. \\
 &- 2g_{\nu\sigma} \left(\frac{(2(p-q)+t)_\pi}{(p-q+t)^2 - W^2} + \frac{(2p+t)_\pi}{(p+t)^2 - W^2} \right) \left. \right] D_{(0)}^{\sigma\pi}(t) \\
 &+ \frac{(2p-q)_\nu}{(p-q)^2 - p^2} [\Sigma^{(0)}((p-q)^2, W^2) - \Sigma^{(0)}(p^2, W^2)] \left. \right\} .
 \end{aligned}$$

In (43), the $\Sigma^{(0)}(p^2, W^2)$ is the lowest order mass correction to a meson with mass W , the third term is the lowest order four-point part, while the fourth term is the contributions of the transverse three-point vertex. The second term corresponds to the photon vacuum polarization contribution, with the scalar function $K(q^2)$ given, in lowest order perturbation theory, by

$$K(q^2) = \frac{1}{3} \left(\frac{e}{4\pi}\right)^2 q^2 \int_{4m^2}^{\infty} dM^2 \frac{(1-4m^2/M^2)^{3/2}}{M^2(q^2-M^2)}. \quad (44)$$

The evaluation of (43) is so involved that we have relegated it to Appendix B. Here we spell out only the final result (with the cuts understood)

$$\text{Im}\Sigma(p^2, W^2) = \text{Im}\Sigma_f(p^2, W^2) + \pi \left(\frac{e}{4\pi}\right)^2 (3-a) \frac{p^4 - W^4}{p^2} [3\Gamma(2-\ell) \left(\frac{M^2}{W^2}\right)^{2-\ell} + 7], \quad (45)$$

with the limit $\ell \rightarrow 2$ understood. M^2 is an arbitrary constant of dimension L^{-2} , which is introduced to compensate the discrepancy in dimension caused by replacing the four-dimensional space-time integral with a 2ℓ -dimensional one. Note especially that in order to achieve self-consistency we have to stick to the same M^2 in all our related calculations. $\text{Im}\Sigma_f(p^2, W^2)$ is the finite part of $\text{Im}\Sigma(p^2, W^2)$, which is independent of both ℓ and M^2 , and is given by

$$\begin{aligned} \text{Im}\Sigma_f(p^2, W^2) = & \pi \left(\frac{e}{4\pi}\right)^2 (a-3) \frac{p^4 - W^4}{p^2} \\ & - \pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{1}{p^2} \int_{W^2}^{p^2} ds \left\{ \frac{4}{s} [(1+a) + \frac{5}{8} (1-a)^2] (p^2-s)(W^2-s) \right. \\ & + \frac{p^2-s}{2s^2} [3s+(3-2a)p^2] [3s+(3-2a)W^2] \} \\ & - \pi \left(\frac{e}{4\pi}\right)^4 \frac{p^2 - W^2}{4p^2} \left\{ (75p^2 + 143W^2) + \frac{2p^2(5p^2 + 6W^2)}{p^2 - W^2} \ln \frac{p^2}{W^2} \right. \\ & + 4 \frac{(p^2 + 3W^2)^2}{p^2 - W^2} \left[3f\left(\frac{p^2}{W^2}\right) - 2 \ln \frac{p^2}{W^2} \ln \frac{W^2}{p^2 - W^2} \right] \end{aligned}$$

$$\begin{aligned}
& + 8 \frac{p^4 + 14p^2 W^2 + W^4}{p^2} \ln \frac{W^2}{p^2 - W^2} \} + \theta(p^2 - gW^2) I(p^2, W^2) \\
& - \pi \left(\frac{e}{4\pi}\right)^4 (1-a) \cdot (p^2 - W^2) \left\{ \frac{3(p^2 + W^2)}{p^2 - W^2} \ln \frac{p^2}{W^2} \right. \\
& \quad \left. + \left[3 \frac{p^2 - W^2}{p^2} - 6 \frac{p^2 + W^2}{p^2} \ln \frac{p^2}{W^2} \right] \right\} \\
& - \frac{\pi}{30} \left(\frac{e}{4\pi}\right)^4 \cdot \frac{(p^2 - W^2)^3}{m^2 p^2} .
\end{aligned} \tag{46}$$

Renormalization of the Equation

Equation (42) obviously needs renormalization. To provide this we neglect all the terms of orders higher than e^4 on the left-hand-side of (42) and approximate it with the quantity

$$\pi \rho(p^2) [p^2 - m^2 + \text{Re}\Sigma^{(0)}(p^2, p^2) - \text{Re}\Sigma^{(0)}(m^2, m^2)] . \tag{47}$$

In the appendix, $\text{Re}\Sigma^{(0)}(W^2, W^2)$ has been evaluated explicitly,

$$\text{Re}\Sigma^{(0)}(W^2, W^2) = - \left(\frac{e}{4\pi}\right)^2 \cdot W^2 \left[3\Gamma(2-\ell) \left(\frac{M^2}{W^2}\right)^{2-\ell} + 7 \right] , \tag{48}$$

with the same ℓ and M^2 as in (45). Hence

$$\begin{aligned}
& \text{Re}\Sigma^{(0)}(p^2, p^2) - \text{Re}\Sigma^{(0)}(m^2, m^2) \\
& = - \left(\frac{e}{4\pi}\right)^2 \cdot (p^2 - m^2) \left[3\Gamma(2-\ell) \left(\frac{M^2}{W^2}\right)^{2-\ell} + 7 \right] + 3 \left(\frac{e}{4\pi}\right)^2 \cdot p^2 \ln \frac{p^2}{m^2} ,
\end{aligned} \tag{49}$$

with the last term arising because

$$\lim_{\ell \rightarrow 2} \Gamma(2-\ell) \left(\frac{M^2}{p^2}\right)^{2-\ell} = \lim_{\ell \rightarrow 2} \Gamma(2-\ell) \left(\frac{M^2}{m^2}\right)^{2-\ell} + \ln \frac{m^2}{p^2} . \tag{50}$$

Inserting (49) into (47) and isolating the divergence, we arrive at

$$\begin{aligned} \pi\rho(p^2)[p^2-m_0^2+\text{Re}\Sigma(p^2,p^2)] &= \pi\rho(p^2)(p^2-m^2)[1+3(\frac{e}{4\pi})^2 \frac{p^2}{p^2-m^2} \ln \frac{p^2}{m^2}] \\ &+ \pi(\frac{e}{4\pi})^4 (3-a) \cdot \frac{p^2+m^2}{p^2} [3\Gamma(2-\ell)(\frac{M^2}{m^2})^{2-\ell} + 7]. \end{aligned} \quad (51)$$

As far as the right-hand-side of (42) is concerned, we replace $\rho(W^2)$ by $\delta(W^2-m^2)$ to separate the divergence from the finite terms

$$\begin{aligned} \int dW^2 \rho(W^2) \frac{\text{Im}\Sigma(p^2, W^2)}{p^2-W^2} &= \int dW^2 \rho(W^2) \frac{\text{Im}\Sigma_f(p^2, W^2)}{p^2-W^2} \\ &+ \pi(\frac{e}{4\pi})^4 (3-a) \frac{p^2+m^2}{p^2} [3\Gamma(2-\ell)(\frac{M^2}{m^2})^{2-\ell} + 7]. \end{aligned} \quad (52)$$

Both (51) and (52) contain the same infinite terms, therefore the divergences cancel one another and leave us with the finite integral equation

$$\pi\rho(p^2)(p^2-m^2)[1+3(\frac{e}{4\pi})^2 \cdot \frac{p^2}{p^2-m^2} \ln \frac{p^2}{m^2}] = \int dW^2 \rho(W^2) \frac{\text{Im}\Sigma_f(p^2, W^2)}{p^2-W^2}. \quad (53)$$

Because we have employed the same M^2 in all our calculations there is no ambiguity in the finite part of $\text{Im}\Sigma(p^2, W^2)$; hence it is unnecessary to introduce an arbitrary constant into (53) as Parker did.

We want to point out that even going to Feynman gauge, $a=1$, (53) is still different from (29). The reason is clear. In renormalizing (21), (26) was employed, while we have adopted (49) in the derivation of (53). Of course this difference will affect the solution, even in asymptopia.

Infrared and ultra-violet solutions of the equation

A full study of (53) is almost impossible without numerical analysis, but it becomes manageable in both infrared and ultra-violet regions. The infrared case is realized by taking the limit that p^2 goes

to m^2 , where the left-hand-side of (53) reduces to

$$\pi\rho(p^2)(p^2-m^2)[1 + 3(\frac{e}{4\pi})^2] . \quad (54)$$

A careful evaluation reveals that

$$\frac{\text{Im}\Sigma_f(p^2, w^2)}{p^2 - w^2} \rightarrow 2\pi(\frac{e}{4\pi})^2 \cdot (a-3)[1 + 3(\frac{e}{4\pi})^2] , \text{ as } p^2 \rightarrow m^2, w^2 \rightarrow m^2$$

where it has been noted that the photon vacuum polarization contribution and the term $\theta(p^2 - w^2)I(p^2, w^2)$ identically vanish and the $\ln \frac{w^2}{p^2 - w^2}$ kind of divergent terms cancel out among themselves in this region. Therefore

$$\int_m^{p^2} dw^2 \rho(w^2) \frac{\text{Im}\Sigma_f(p^2, w^2)}{p^2 - w^2} \rightarrow 2\pi(\frac{e}{4\pi})^2 \cdot (a-3)[1 + 3(\frac{e}{4\pi})^2] \int_m^{p^2} dw^2 \rho(w^2) ,$$

as $p^2 \rightarrow m^2$. (55)

Combining (54) with (55) results in the familiar equation

$$\rho(p^2)(p^2 - m^2) = 2(\frac{e}{4\pi})^2 (a-3) \int_m^{p^2} dw^2 \rho(w^2) , \quad (56)$$

of which the solution is

$$\rho(p^2) \approx \left(\frac{1}{p^2 - m^2} \right)^{1 - 2(a-3)(\frac{e}{4\pi})^2} . \quad (57)$$

As we expected, our equation produces the standard infrared behaviour.

This shows that the inclusion of transverse vertices into gauge technique does not affect the infrared solution of the spectral function.

Turning to the ultra-violet domain, $\frac{p^2}{m^2} \rightarrow +\infty$, we achieve asymptotic self-consistency with

$$\rho(p^2) \approx (p^2)^{-1 + (\frac{e}{4\pi})^2 (a-3)} [1 - 3(\frac{e}{4\pi})^2 \ln \frac{p^2}{m^2}] ,$$

in the small coupling limit. The logarithmic term will certainly be affected by going to higher order gauge approximations, and we can not

even hazard a guess about what will happen when we sum all the subdominant terms.

IV.4 GAUGE INVARIANCE OF THE PHOTON VACUUM POLARIZATION TENSOR

As with spinor electrodynamics, the vacuum polarization tensor which, in the case of scalar electrodynamics reads

$$\begin{aligned} \pi_{\mu\nu}(t) = & -Z_\phi ie^2 \int \bar{d}^4 p (2p+t)_\mu G_\nu(p, p+t) + 2Z_\phi ie^2 \int \bar{d}^4 p g_{\mu\nu} \Delta(p) \\ & + 2Z_\phi e^4 \int \bar{d}^4 p \bar{d}^4 t' D_{\mu\rho}(t') G_{\nu\sigma}(p, p+t+t'; -t, -t') g^{\rho\sigma}, \end{aligned} \quad (58)$$

should not only be transverse but also invariant under gauge transformations. In particular, in covariant gauges it should be independent of the gauge parameter 'a'.

However, the lowest order gauge technique meets this requirement only to the e^2 order; to order e^4 and higher the $\pi_{\mu\nu}$, although transverse, does depend on the gauge parameter a . With the refined G_μ given in (34), one may hope the gauge independence of (58) is rectified. We will verify this to e^4 order explicitly. Let

$$G_\mu(p', p) = \int dW^2 \rho(W^2) \frac{1}{p'^2 - W^2} [(p'+p)_\mu + \lambda_\mu(p', p|W)] \frac{1}{p^2 - W^2}, \quad (59)$$

$$\text{and } G_{\mu\nu}^{(0)}(p', p; t, p'-p-t) = \int dW^2 \rho(W^2) \frac{1}{p'^2 - W^2} g_{\mu\nu}^{(0)}(p', p; t, p'-p-t|W) \frac{1}{p^2 - W^2}.$$

Then we can express the approximate vacuum polarization tensor by

$$\begin{aligned} \pi_{\mu\nu}(t) = & Z_\phi \int dW^2 \rho(W^2) \left\{ -ie^2 \int \bar{d}^4 p \frac{(2p+t)_\mu}{p^2 - W^2} [(2p+t)_\nu + \lambda_\nu(p, p+t|W)] \frac{1}{(p+t)^2 - W^2} \right. \\ & + 2ie^2 \int \bar{d}^4 p \frac{g_{\mu\nu}}{p^2 - W^2} + 2e^4 \int \bar{d}^4 p \bar{d}^4 t' D_{(0)\mu\rho}(t') \frac{1}{p^2 - W^2} g_{\nu\rho}^{(0)}(p, p+t+t'; -t, -t'|W) \\ & \left. \times \frac{g^{\rho\rho'}}{(p+t+t')^2 - W^2} \right\}. \end{aligned} \quad (60)$$

A full analysis of (60) is very difficult because of the lack of an exact $\rho(W^2)$ to e^4 order, and also because λ_μ and $g_{\mu\nu}^{(0)}$ are quite complicated; therefore certain reference to perturbation theory is required.

To e^4 order, it is necessary to take the lowest order approximation of both Z_ϕ and $\rho(W^2)$ in the terms connected with λ_ν and $g_{\mu\nu}^{(0)}(p, p+t+t'; -t, -t'|W)$, while in the other terms we need only consider the first order approximations to them. This leaves us with

$$\begin{aligned} \pi_{\mu\nu}(t) = & Z_\phi \pi_{\mu\nu}^{(0)}(k|m) + e^2 \int dW^2 \rho_1(W^2) \pi_{\mu\nu}^{(0)}(t|W) \\ & - ie^2 \int d^4 p (2p+t)_\mu \frac{1}{p^2 - m^2} \lambda_\nu(p, p+t|m) \frac{1}{(p+t)^2 - m^2} \\ & + 2e^4 \int d^4 p d^4 t' D_{(0)\mu\rho}(t') \frac{g^{\rho\rho'}}{p^2 - m^2} g_{\nu\rho'}^{(0)}(p, p+t+t'; -t, -t'|m) \frac{1}{(p+t+t')^2 - m^2}, \end{aligned} \quad (61)$$

with $\pi_{\mu\nu}^{(0)}(t|W)$ the lowest order vacuum polarization tensor, given by

$$\pi_{\mu\nu}^{(0)}(t|W) = -ie^2 \int d^4 p \frac{(2p+t)_\mu (2p+t)_\nu}{(p^2 - W^2)[(p+t)^2 - W^2]} + 2ie^2 \int d^4 p \frac{g_{\mu\nu}}{p^2 - W^2}. \quad (62)$$

(61) is not manifestly gauge invariant owing to the appearance of the gauge dependent quantities in it. To explore its gauge properties, we differentiate it with respect to a ,

$$\begin{aligned} \frac{\partial \pi_{\mu\nu}(k)}{\partial a} = & \frac{\partial Z_\phi}{\partial a} \pi_{\mu\nu}^{(0)}(t|m) + \int dW^2 \frac{\partial e^2 \rho_1(W^2)}{\partial a} \pi_{\mu\nu}^{(0)}(t|W) \\ & - ie^2 \int d^4 p \frac{(2p+t)_\mu}{p^2 - m^2} \frac{\partial}{\partial a} \lambda_\nu(p, p+t|m) \frac{1}{(p+t)^2 - m^2} \\ & - 2e^4 \int d^4 t' d^4 p \frac{1}{t'^4} \cdot \frac{1}{p^2 - m^2} \cdot \frac{1}{(p+t+t')^2 - m^2} t'_\mu t'^\rho g_{\nu\rho}^{(0)}(p, p+t+t'; -t, -t'|m). \end{aligned} \quad (63)$$

In (63)

$$\frac{\partial e^{2\rho_1(W^2)}}{\partial a} = \frac{\partial}{\partial a} \frac{\text{Im}\Sigma^{(0)}(W^2, m^2)}{\pi(W^2 - m^2)^2} = \left(\frac{e}{4\pi}\right)^2 \frac{W^2 + m^2}{W^2(W^2 - m^2)} \quad (64)$$

$$\frac{\partial Z_\phi}{\partial a} = - \frac{\partial^2 \text{Re}\Sigma^{(0)}(W^2, m^2)}{\partial W^2 \partial a} \Big|_{W^2 = m^2} \quad (65)$$

where the following relationship has been used

$$Z_\phi^{-1} = \int dW^2 \rho(W^2) \approx 1 + \frac{\partial \text{Re}\Sigma^{(0)}(W^2, m^2)}{\partial W^2} \Big|_{W^2 = m^2}$$

The original definition of $\Sigma^{(0)}(W^2, m^2)$, that is, (36), leads us to

$$\frac{\partial^2 \text{Re}\Sigma^{(0)}(W^2, m^2)}{\partial W^2 \partial a} = -ie^2 \int d^4k \frac{1}{k^4} \quad (66)$$

which is both infrared and ultra-violet divergent. To avoid ambiguity, we will keep the expression (66) intact and work with it later.

The last term of (63) can be evaluated almost immediately. Using the definition of $g_{\mu\nu}^{(0)}$, and repeatedly utilizing the identity

$$\frac{A-B}{A \cdot B} = \frac{1}{B} - \frac{1}{A}$$

we obtain

$$\begin{aligned} & -2e^4 \int d^4p d^4t' \frac{1}{t'^4} \frac{1}{p^2 - m^2} \cdot \frac{1}{(p+t+t')^2 - m^2} t'_\mu g_{\nu\rho}^{(0)}(p, p+t+t'; -t, -t'|m) t'^\rho \\ & = -2e^4 \int d^4p d^4t' \frac{t'_\mu}{t'^4} \cdot \frac{[2(p+t') + t]_\nu}{[(p+t')^2 - m^2][(p+t+t')^2 - m^2]} \end{aligned} \quad (67)$$

The evaluation of the second term requires special care, owing to the existence of both linear and quadratic divergences. We work out the derivative first:

$$\begin{aligned}
\frac{\partial \lambda_\mu(p, p' | m)}{\partial a} = & - \frac{(p' + p)_\mu}{\pi} \int dW^2 \frac{\partial}{\partial a} \text{Im} \Sigma^{(0)}(W^2, m^2) \left\{ \frac{1}{(p^2 - W^2)(p'^2 - W^2)} \right. \\
& + \frac{1}{W^2 - m^2} \left(\frac{1}{p^2 - W^2} + \frac{1}{p'^2 - W^2} \right) + \frac{1}{(W^2 - m^2)^2} \Big\} \\
& - ie^2 \int d^4 t' \frac{1}{t'^4} (p + p' + 2t')_\mu \frac{(p^2 - m^2)(p'^2 - m^2)}{[(p + t')^2 - m^2][(p' + t')^2 - m^2]} .
\end{aligned} \tag{68}$$

Now substituting (68) in the second term of (63), we arrive at

$$\begin{aligned}
& - ie^2 \int d^4 p \frac{1}{p^2 - m^2} (2p + t)_\mu \frac{\partial \lambda_\nu(p, p + t | m)}{\partial a} \frac{1}{(p + t)^2 - m^2} \\
& = - \frac{1}{\pi} \int dW^2 \frac{\partial \text{Im} \Sigma^{(0)}(W^2, m^2)}{\partial a} \cdot \frac{1}{(W^2 - m^2)^2} \cdot (-ie^2) \int d^4 p \frac{(2p + t)_\mu (2p + t)_\nu}{(p^2 - W^2)[(p + t)^2 - W^2]} \\
& - ie^2 (-ie^2) \int d^4 p d^4 t' \frac{1}{t'^4} \frac{(2p + t)_\mu [2(p + t') + t]_\nu}{[(p + t')^2 - m^2][(p + t' + t)^2 - m^2]} .
\end{aligned} \tag{69}$$

Adding (67) and (69), we end up with

$$\begin{aligned}
& - \frac{1}{\pi} \int dW^2 \frac{\partial \text{Im} \Sigma^{(0)}(W^2, m^2)}{\partial a} \frac{1}{(W^2 - m^2)^2} (-ie^2) \int d^4 p \frac{(2p + t)_\mu (2p + t)_\nu}{(p^2 - W^2)[(p + t)^2 - W^2]} \\
& - ie^2 (-ie^2) \int d^4 p d^4 t' \frac{1}{t'^4} \cdot \frac{[2(p + t') + t]_\mu [2(p + t') + t]_\nu}{[(p + t')^2 - m^2][(p + t' + t)^2 - m^2]} .
\end{aligned} \tag{70}$$

We are not allowed to translate variables in the second term (which is quadratically divergent) without taking into account the consequent surface terms. Of this statement, a convincing illustration is provided by the observation that shifting p to $p - t'$ violates the transversal property of (70) with respect to t'^ν . To get around the "surface term" problem, we insert the following quantity in (70)

$$\begin{aligned} & \frac{1}{\pi} \int dW^2 \frac{\partial \text{Im}\Sigma^{(0)}(W^2, m^2)}{\partial a} \frac{1}{(W^2 - m^2)^2} (-ie^2) \int \bar{d}^4 p \frac{2g_{\mu\nu}}{p^2 - W^2} \\ & + ie^2 (-ie^2) \int \bar{d}^4 p \bar{d}^4 t' \frac{2g_{\mu\nu}}{t'^4 [(p+t')^2 - m^2]} . \end{aligned} \quad (71)$$

(71) identically vanishes as can be easily checked out by noting that

$$\text{Im}[i \int \bar{d}^4 t' \frac{1}{t'^4} \cdot \frac{1}{(p+t')^2 - m^2}] = \frac{\partial}{\partial a} \frac{\text{Im}\Sigma^{(0)}(p^2, m^2)}{(p^2 - m^2)^2} .$$

Thus we arrive at

$$\begin{aligned} (70) = & - \frac{1}{\pi} \int dW^2 \frac{\partial}{\partial a} \frac{\text{Im}\Sigma^{(0)}(W^2, m^2)}{(W^2 - m^2)^2} (-ie^2) \int \bar{d}^4 p \left\{ \frac{(2p+t)_\mu (2p+t)_\nu}{(p^2 - W^2)[(p+t)^2 - W^2]} - \frac{2g_{\mu\nu}}{p^2 - W^2} \right\} \\ & - ie^2 (-ie^2) \int \bar{d}^4 p \bar{d}^4 t' \frac{1}{t'^4} \left\{ \frac{[2(p+t')+t]_\mu [2(p+t')+t]_\nu}{[(p+t')^2 - m^2][(p+t+t')^2 - m^2]} - \frac{2g_{\mu\nu}}{(p+t')^2 - m^2} \right\} . \end{aligned} \quad (72)$$

Dimensionally regularizing (72), translating variable in the p integration of the second term, and recalling the definition of $\pi_{\mu\nu}^{(0)}(t|W)$, we find

$$(70) = - \frac{1}{\pi} \int dW^2 \frac{\partial}{\partial a} \frac{\text{Im}\Sigma^{(0)}(W^2, p^2)}{(W^2 - m^2)^2} \pi_{\mu\nu}^{(0)}(t|W) - ie^2 \int \bar{d}^4 t' \frac{1}{t'^4} \cdot \pi_{\mu\nu}^{(0)}(t|m) . \quad (73)$$

Inserting (73) into (63) and making use of the relations (64)-(66) leads us to

$$\frac{\partial \pi_{\mu\nu}^{(0)}(t)}{\partial a} \equiv 0 . \quad (74)$$

We conclude that the inclusion of transverse vertices improves the gauge property of the vacuum polarization. At least to e^4 order, the vacuum polarization tensor is still gauge independent in covariant gauges.

IV.5 THE LOWEST ORDER GAUGE TECHNIQUE IN THE AXIAL GAUGE

To complete our discussion of scalar electrodynamics we go to the axial gauge specified by

$$n^\mu A_\mu(x) = 0, \quad n^2 \neq 0 \quad (75)$$

in order to investigate the properties of the spectral function using the lowest order gauge technique. As pointed out in chapter II, the Dyson-Schwinger equations and Ward-Green-Takahashi identities in this gauge are of the same forms as those in covariant gauges, but the spectral representation of the meson propagator becomes

$$\Delta(p) = \int dW^2 \frac{\rho(W^2, p \cdot n)}{p^2 - W^2 + i0^+} \quad (76)$$

with the spectral function depending on both the external vector n and the momentum p . This greatly complicates our problem because the approximate three-point photon amputated Green function can no longer be expressed in such a tidy form as (3), and the mass correction also depends on the non-covariant argument $p \cdot n$. This obstacle prevented people developing the gauge technique to the next level; even in the lowest order, the problem has only been investigated in the infrared region. In this section, we will review the work of Delbourgo and Phocas-Cosmetatos [39] and then solve the gauge technique equation appropriate to the axial gauge at all momenta.

The authors inserted the following ansatz [38],

$$\begin{aligned} G_\mu^{(0)}(p', p) = & \frac{1}{2} \int dW^2 \frac{1}{p'^2 - W^2} (p' + p)_\mu \frac{1}{p^2 - W^2} [\rho(W^2, p' \cdot n) + \rho(W^2, p \cdot n)] \\ & - \frac{1}{2} \int dW^2 \left(\frac{1}{p'^2 - W^2} + \frac{1}{p^2 - W^2} \right) n_\mu \frac{\rho(W^2, p' \cdot n) - \rho(W^2, p \cdot n)}{p' \cdot n - p \cdot n} \end{aligned} \quad (77)$$

in the Dyson-Schwinger equation

$$Z_{\phi}^{-1} = \Delta(p)(p^2 - m_0^2) - ie^2 \int d^4k (2p-k)_{\mu} D_{(0)}^{\mu\nu}(k) G_{\nu}^{(0)}(p-k, p; k) + O(e^4) \quad (78)$$

where

$$D_{(0)}^{\mu\nu}(k) = (-g^{\mu\nu} + \frac{n_{\mu} k_{\nu} + n_{\nu} k_{\mu}}{n \cdot k} - \frac{n^2 k_{\mu} k_{\nu}}{(n \cdot k)^2}) \frac{1}{k^2 + i0^+}$$

and dropped all the higher order terms. They obtained

$$Z_{\phi}^{-1} = \int dW^2 \frac{1}{p^2 - W^2} \{ (p^2 - m_0^2) \rho(W^2, p_0) - \frac{i}{2} e^2 \int d^4k \frac{(2p-k)_i (\delta_{ij} - k_i k_j / k_0^2) (2p-k)_j}{k^2 [(p-k)^2 - W^2]} [\rho(W^2, p_0) + \rho(W^2, p_0 - k_0)] \} \quad (79)$$

upon taking $n=(1,0,0,0)$. By replacing the lowest order self-energy correction appropriate to a scalar of mass W in the axial gauge by

$$\pi(p^2, p_0, W^2) = \int dk_0 \pi_{k_0}(p^2, p_0, W^2) \quad (80)$$

(79) was rewritten as

$$Z_{\phi}^{-1} = \int dW^2 \frac{1}{p^2 - W^2} \{ (p^2 - m_0^2) \rho(W^2, p_0) + \frac{1}{2} \int dk_0 \pi_{k_0}(p^2, p_0, W^2) [\rho(W^2, p_0) + \rho(W^2, p_0 - k_0)] \} \quad (81)$$

The imaginary part of (81) gave

$$(p^2 - m^2) \rho(p^2, p_0) = \frac{1}{2\pi} \int_m^{p^2} dW^2 \int dk_0 [\rho(W^2, p_0) + \rho(W^2, p_0 - k_0)] \frac{\text{Im} \pi_{k_0}(p^2, p_0, W^2)}{p^2 - W^2} \quad (82)$$

after necessary renormalizations were carried out. $\text{Im} \pi_{k_0}$ was worked out explicitly, it read

$$\text{Im} \pi_{k_0}(p^2, p_0, W^2) = 4\pi \left(\frac{e}{4\pi}\right)^2 |\vec{p}| \cdot \left[1 - \frac{2p_0 k_0 - p^2 + W^2}{2|\vec{p}| k_0} \right] \theta(p^2 - W^2) \quad (83)$$

and the integration domain of (80) was specified as $\left(\frac{p^2 - W^2}{2(p_0 + |\vec{p}|)}, \frac{p^2 - W^2}{2(p_0 - |\vec{p}|)} \right)$.

By changing integration variable, (82) was simplified to

$$(p^2 - m^2) \rho(p^2, p_0) = \left(\frac{e}{4\pi}\right)^2 \int_m^2 dW^2 \int_{-1}^{+1} du \frac{1-u^2}{\left(u - \frac{p_0}{|\vec{p}|}\right)^2} [\rho(W^2, p_0) + \rho(W^2, p_0 - \frac{p^2 - W^2}{2(p_0 - u|\vec{p}|})] . \quad (84)$$

Due to the complicated u dependence entering the non-covariant argument of ρ , (84) was only solved in the infrared, producing the solution

$$\rho(p^2, p.n) \approx (p^2 - m^2)^{-1} e^2 \cdot (-1 + b^{-1} \tan b^{-1}) / 2\pi^2, \quad p^2 \rightarrow m^2, \quad (85)$$

where b is given by

$$b^{-1} \tan b^{-1} - 1 = \int_0^1 du [1 + u^2 \left(\frac{n^2 p^2}{(p.n)^2} - 1 \right)]^{-1} . \quad (86)$$

It is not impossible to solve (84) completely. To do this, we cast (84) into a more general form; for an arbitrary time-like gauge vector $n(n^2=1)$, (84) becomes

$$(p^2 - m^2) \rho(p^2, p.n) = \left(\frac{e}{4\pi}\right)^2 \int_m^2 dW^2 \int_{-1}^{+1} du \frac{1-u^2}{\left(u - \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right)^2} \cdot [\rho(W^2, p.n) + \rho(W^2, p.n - \frac{(1 - \frac{W^2}{p^2}) \sqrt{p^2}}{2(\gamma - u\sqrt{\gamma^2 - 1})})] \quad (87)$$

with

$$\gamma = \frac{(p.n)^2}{p^2 \cdot n^2} \quad (88)$$

being a parameter independent of the scale of p . We should note that $\gamma \geq 1$ is always true. This guarantees the reality of the factor $\sqrt{\gamma^2 - 1}$ appearing in (87). Recognizing the important fact that the non-covariant argument of ρ is accompanied by an e^2 factor, we see that a partial

differentiation with respect to this argument will bring ρ to e^2 order, that is

$$\frac{\partial \rho(W^2, p.n)}{\partial (p.n)} \sim 0(e^2) \quad . \quad (89)$$

This can be checked straight-forwardly in perturbation theory.

Therefore, upon neglecting the higher order terms we can convert (87) to

$$(p^2 - m^2) \frac{\partial \rho(p^2, p.n)}{\partial p^2} + \rho(p^2, p.n) = 2\left(\frac{e}{4\pi}\right)^2 \int_{-1}^{+1} du \frac{1-u^2}{\left(u - \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right)^2} \rho(p^2, p.n) \quad (90)$$

by differentiating (87) once with respect, to p^2 . Working out the integral on the right-hand-side and solving this equation we obtain

$$\rho(p^2, p.n) = R(e^2, \gamma) (p^2 - m^2)^{-1+e^2(-1+b^{-1}\tan b^{-1})/2\pi^2} \quad (91)$$

$R(e^2, \gamma)$ is an arbitrary function of e^2 and γ only such that as $e^2 \rightarrow 0$, $R \rightarrow 1$. In the infrared limit, (91) coincides with (85).

This leads us to conjecture that in the lowest order gauge approximation, the spectral function $\rho(W^2, p.n)$ assumes the following expression

$$\rho(W^2, p.n) = R(e^2, \gamma) (W^2 - m^2)^{-1+e^2(-1+b^{-1}\tan b^{-1})/2\pi^2} \quad (92)$$

It should be possible to prove this conjecture, at least to order e^4 in perturbation theory, but we have not done so.

V. GAUGE TECHNIQUE AND SCHWINGER MODEL

Often one can exactly solve theoretical models in two-dimensions, although these models can only be studied perturbatively in four-dimensions. An examination of their properties in two dimensions provides us with some useful insights about their behaviour in the real world. This chapter is devoted to the studies of two-dimensional models (particularly the massless Schwinger model), through the gauge technique. Due to the fact that the gauge technique equations for the spectral functions are usually too difficult to solve exactly in momentum space, we will slightly modify the conventional method; thus we abandon the spectral representations for the two-point Green functions and solve the Dyson-Schwinger equations for particle propagators directly in the configuration space. We are also able to extend our work to finite temperature.

V.1 SCHWINGER MODEL [19]

The massless Schwinger model consists of a massless spinor meson coupled to a vector meson by minimal interaction. Denoting the spinor field by $\psi(x)$, and the vector field by $B_\mu(x)$, the Lagrangian reads

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \bar{\psi}(x) \gamma^\mu (i\partial_\mu - gB_\mu(x)) \psi(x) , \quad (1)$$

with

$$F_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) .$$

$\mathcal{L}(x)$ is an invariant under both the normal gauge transformation

$$B_\mu(x) \rightarrow B'_\mu(x) = B_\mu(x) + \frac{1}{g} \partial_\mu \theta(x) , \quad \psi(x) \rightarrow \psi'(x) = e^{-i\theta(x)} \psi(x) ,$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) e^{i\theta(x)} , \quad (2)$$

and the chiral transformation

$$B_\mu(x) \rightarrow B'_\mu(x) = B_\mu(x) - \frac{i}{g} \epsilon_{\mu\nu} \partial^\nu \theta_5(x), \quad \psi(x) \rightarrow \psi'(x) = e^{i\gamma_5 \theta_5(x)} \psi(x),$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) e^{i\gamma_5 \theta_5(x)}, \quad (3)$$

with $\gamma_5 = -i\gamma^0\gamma^1$. This fact manifests itself in two sets of gauge identities, namely, the vector and axial identities, of which the first two assume the forms [44]

$$k^\mu G_\mu(p, p-k) = S(p-k) - S(p), \quad (4)$$

$$ik_\nu \epsilon^{\mu\nu} G_\mu(p, p-k) = S(p)\gamma_5 + \gamma_5 S(p-k). \quad (5)$$

Observe that the vector identity (4) completely determines the longitudinal part of the three-point amplitude, while the axial identity (5) contains entirely the transversal part of the amplitude. Therefore they permit the exact solution of G_μ , in terms of the spinor meson propagator S ,

$$\text{i.e. } G_\mu(p, p-k) = \frac{k_\mu}{k^2} [S(p-k) - S(p)] + \frac{i\epsilon_{\mu\nu} k^\nu}{k^2} [S(p)\gamma_5 + \gamma_5 S(p-k)]. \quad (6)$$

This result was first obtained by Delbourgo and Thompson [44]. In covariant gauges, by using the spectral representation for spinor meson propagator,

$$S(p) = \int dW \frac{\rho(W)}{\not{p} - W + i\epsilon(W)},$$

(6) reduces to

$$G_\mu(p, p-k) = \int dW \frac{\rho(W)}{\not{p} - W} \left[\gamma_\mu - \frac{i2W\gamma_5 \epsilon_{\mu\nu} k^\nu}{k^2} \right] \frac{1}{\not{p} - k - W}. \quad (7)$$

This is the spectral representation of the amplitude first conjectured by Gardner [43].

To analyse this model with the gauge technique, we also need Dyson-Schwinger equations. From the Lagrangian (1) we can easily obtain the equations for the spinor and vector meson propagators

respectively

$$1 = \not{S}(p) - ig^2 \int \bar{d}^2 k \gamma_\nu G_\mu(p+k, p) D^{\mu\nu}(k) \quad , \quad (8)$$

$$D_{\mu\nu}^{-1}(k) = D_{\mu\nu}^{(0)-1}(k) + ig^2 \int \bar{d}^2 p \text{tr}[G_\mu(p, p-k) \gamma_\nu] \quad , \quad (9)$$

with $D_{\mu\nu}^{(0)-1}(k)$ the inverse bare vector meson propagator. If we specify a covariant gauge through the gauge fixing term

$$\mathcal{L}_a = \frac{1}{2} (\partial \cdot A) \left[\frac{\mu^2}{2} + \frac{1}{a} \right] (\partial \cdot A) \quad , \quad (10)$$

the inverse bare vector meson propagator reads

$$D_{\mu\nu}^{(0)-1}(k) = - (k^2 \eta_{\mu\nu} - k_\mu k_\nu) - \frac{k_\mu k_\nu}{k^2} (\mu^2 - \frac{k^2}{a}) \quad (11)$$

where we define $\mu^2 = \frac{g^2}{\pi}$, and $\eta_{\mu\nu}$ is the metric tensor,

$$\eta_{00} = - \eta_{11} = 1 \quad , \quad \eta_{01} = \eta_{10} = 0 \quad . \quad (12)$$

(Instead, for axial gauges, $D_{\mu\nu}^{(0)-1}(k)$ is given by (II-105) .)

Substituting (7) into (9) and using the sum rule

$$\int dW \rho(W) = 1$$

one can easily show that

$$\begin{aligned} \pi_{\mu\nu}(k) &= ig^2 \int \bar{d}^2 p \text{tr}[G_\mu(p, p-k) \gamma_\nu] \\ &= \mu^2 (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \int dW \rho(W) \\ &= \mu^2 (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \quad . \end{aligned} \quad (13)$$

Therefore gauge symmetry is spontaneously broken with the vector meson acquiring a dynamical mass $\mu = \frac{g}{\sqrt{\pi}}$. With (13) the full propagator can be worked out straight-forwardly,

$$D_{\mu\nu}(k) = \frac{-1}{k^2 - a^2} (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) - a \frac{k_\mu k_\nu}{k^2(k^2 - a^2)} . \quad (14)$$

By substituting (14) and (7) into (8), one will arrive at the conventional gauge technique equation for the spectral function $\rho(W)$. But unfortunately the resultant equation is usually very difficult to solve, except in the Landau gauge ($a=1$), where $\rho(W) = \delta(W)$ [42]. To substantiate our statement above, we spell out the equations in the Feynman gauge ($a=0$) [86],

$$\rho_+(W) = \frac{-1}{2W} \int_0^W d\beta \{ \rho_+(\beta) - \rho_+[(\beta^2 - \frac{\mu^2 \beta^2}{W^2 - \beta^2})^{\frac{1}{2}}] \theta(\beta^2 - \frac{\mu^2 \beta^2}{W^2 - \beta^2}) \} , \quad \int dW \rho_+(W) = 1, \quad (15)$$

$$\rho_-(W) = \frac{1}{2W^2} \int_0^W d\beta \{ \beta \rho_-(\beta) - (1 - \frac{\mu^2}{W^2 - \beta^2})^{-\frac{1}{2}} \rho_-[(\beta^2 - \frac{\mu^2 \beta^2}{W^2 - \beta^2})^{\frac{1}{2}}] \theta(\beta^2 - \frac{\mu^2 \beta^2}{W^2 - \beta^2}) \} . \quad (16)$$

Where $\rho_\pm = \frac{1}{2}[\rho(W) \pm \rho(-W)]$, $W \geq 0$. Except at asymptopia, it is impossible to solve these equations exactly. In non-covariant gauges the situation becomes even worse, because the spectral function in general also depends on the non-covariant gauge parameters.

V.2 EXACT SOLUTIONS OF THE SCHWINGER MODEL

In this section we propose a variant of the gauge technique to solve the Schwinger model. Inserting (6) in (8) and noting that $D_{\mu\nu}(k)$ is an even function of k , we arrive at

$$1 = \not{p} S(p) - ig^2 \int \bar{d}^2 k \frac{1}{k^2} \gamma_\nu S(p+k) [-k_\mu + i\epsilon_{\mu\sigma} k^\sigma \gamma_5] D^{\mu\nu}(k) . \quad (17)$$

Let us denote the spinor meson propagator in the configuration space by $S(x)$ and convert (17) into this space, we end up with the following equation

$$\delta(x) = i\not{x}S(x) - ig^2 \int \bar{d}^2k \frac{1}{k^2} \gamma_\nu S(x) [-k_\mu + i\epsilon_{\mu\sigma} k^\sigma \gamma_5] D^{\mu\nu}(k) e^{ikx} . \quad (18)$$

Look for a parity conserving solution:

$$S_A(x) = S_0(x) \cdot \exp(Q(x)) , \quad (19)$$

with $S_0(x)$ the bare propagator defined by

$$i\not{x}S_0(x) = \delta(x) , \quad S_0(x) = \frac{1}{2\pi(x \cdot \gamma + i\epsilon)} , \quad (20)$$

and $Q(x)$ satisfying

$$[Q(x), \gamma_5] = 0 , \quad Q(0) = 0 . \quad (21)$$

Insert (19) into (18). After carrying out some elementary manipulations, the result is

$$0 = i\gamma^\mu S_0(x) \partial_\mu Q(x) - ig^2 \int \bar{d}^2k \frac{1}{k^2} \gamma_\nu S_0(x) (-k_\mu + i\epsilon_{\mu\sigma} k^\sigma \gamma_5) D^{\mu\nu}(k) e^{ikx} . \quad (22)$$

Because of the first identity of (21), $Q(x)$ can only consist of terms with even numbers of γ -matrices. In two dimensions, these terms are scalar functions or some scalar functions accompanied with a γ_5 , that is

$$Q(x) = f(x) + \gamma_5 g(x) . \quad (23)$$

Multiplying (22) with $2\pi x \cdot \gamma$ and utilizing (23) leads us to

$$\begin{aligned} & i\not{x}\tilde{Q}(x) + ig^2 \int \bar{d}^2k \frac{1}{k^2} \gamma_\mu (k_\nu + i\epsilon_{\nu\sigma} k^\sigma \gamma_5) D^{\mu\nu}(k) e^{ikx} \\ & = i\not{x}\tilde{Q}(x) + ig^2 \int \bar{d}^2k \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) e^{ikx} = 0 \end{aligned} \quad (24)$$

where $\tilde{Q}(x) = f(x) - \gamma_5 g(x) = S_0(x) Q(x) \cdot 2\pi x \cdot \gamma . \quad (25)$

The solution of (24) which satisfies the boundary condition given in (21) can be obtained straight-forwardly. It reads

$$\tilde{Q}(x) = ig^2 \int \bar{d}^2k \frac{1}{k} \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) (e^{ikx} - 1) . \quad (26)$$

This in turn provides the solution for $S_A(x)$:

$$S_A(x) = S_0(x) \exp\left[\frac{ig^2}{x \cdot \gamma} \int d^2k \frac{1}{k} \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) (e^{ikx} - 1)_{x \cdot \gamma}\right] . \quad (27)$$

However, $S_A(x)$ is not necessarily the unique solution to (18). In fact other solutions can be obtained by letting

$$S(x) = S_A(x) + S_S(x) \quad (28)$$

with

$$[S_S(x), \gamma_5] = 0 , \quad (29)$$

and solving (18) for $S_S(x)$. Inserting (28) into (18) leads to

$$0 = i \not{\partial} S_S(x) + ig^2 \int d^2k \frac{D^\mu_\mu(k) k}{k^2} e^{ikx} S_S(x) . \quad (30)$$

Where the fact that $D_{\mu\nu}(k)$ is symmetric in the indices μ and ν has been used to tidy up the expression of (30). Obviously the solution of (30) is

$$S_S(x) = C \cdot \exp\left(ig^2 \int d^2k \frac{D^\mu_\mu(k)}{k^2} e^{ikx}\right) , \quad (31)$$

with C an arbitrary constant (and possibly infinite in order to make $S_S(x)$ finite). However, we are not able to determine the constant within the framework of the gauge technique.⁺

Next we apply these solutions to study the vector meson vacuum polarization. Denoting the momentum versions of $S_S(x)$ and $S_A(x)$, respectively, by $S_S(p)$ and $S_A(p)$, with

⁺ Our results are compatible with the conventional gauge technique result, say equation (15) and (16) in the Feynman gauge. There the odd part of the spectral function, $\rho(W)$, is given by the homogeneous equation (16) and can therefore only be determined up to an overall constant factor.

$$S_A(p) = \int d^2\ell \frac{1}{\not{p}-\not{\ell}+i\epsilon} F(\ell) \quad , \quad F(\ell) = \int d^2x \exp(Q(x)) e^{-i\ell x} \quad ,$$

then inserting them into (6), we arrive at

$$\begin{aligned} G_{\mu}(p, p-k) &= \int d^2\ell \frac{1}{k^2} \{ k_{\mu} \left[\frac{1}{\not{p}-\not{\ell}-\not{k}+i\epsilon} - \frac{1}{\not{p}-\not{\ell}+i\epsilon} \right] + i\epsilon_{\mu\nu} k^{\nu} \left[\frac{1}{\not{p}-\not{\ell}+i\epsilon} \gamma_5 + \right. \\ &\quad \left. \gamma_5 \frac{1}{\not{p}-\not{\ell}-\not{k}+i\epsilon} \right] \} F(\ell) \\ &+ \frac{1}{k^2} \{ k_{\mu} [S_S(p-k) - S_S(p)] + i\epsilon_{\mu\nu} k^{\nu} \gamma_5 [S_S(p-k) + S_S(p)] \} \\ &= \int d^2\ell \frac{1}{\not{p}-\not{\ell}+i\epsilon} \gamma_{\mu} \frac{1}{\not{p}-\not{\ell}-\not{k}+i\epsilon} F(\ell) \\ &+ \frac{1}{k^2} \{ k_{\mu} [S_S(p-k) - S_S(p)] + i\epsilon_{\mu\nu} k^{\nu} \gamma_5 [S_S(p-k) + S_S(p)] \} . \end{aligned} \quad (32)$$

Substituting (32) into the vacuum polarization tensor and noting that the second part does not contribute, we obtain

$$\pi_{\mu\nu}(k) = ig^2 \int d^2\ell \text{ptr} \left[\int d^2\ell \frac{1}{\not{p}-\not{\ell}} \gamma_{\mu} \frac{1}{\not{p}-\not{k}-\not{\ell}} F(\ell) \gamma_{\nu} \right] . \quad (33)$$

Upon translating variables and using the fact

$$\int d^2\ell F(\ell) = \exp(Q(0)) = 1 \quad ,$$

(33) can be simplified to

$$\pi_{\mu\nu}(k) = ig^2 \int d^4\text{ptr} \left[\frac{1}{\not{p}} \gamma_{\mu} \frac{1}{\not{p}-\not{k}} \gamma_{\nu} \right] . \quad (34)$$

The integration can be easily performed, and we arrive at

$$\pi_{\mu\nu}(k) = \mu^2 \left(\eta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right) . \quad (35)$$

This gives a dynamical vector meson mass $\frac{g}{\sqrt{\pi}}$.

We wish to emphasize that our arguments above are not subject to gauge choice, therefore the solutions (27), (31) and (35) are valid in

all linear gauges of the Schwinger model. Moreover, they are also true for the Thirring model [20]. Although that model is not a gauge theory, the phase and chiral transformations still provides us with vector and axial gauge identities of exactly the same forms as (4) and (5); hence the above analysis is also applicable to it, except that we have to incorporate a renormalization constant to the spinor meson propagator. We will not investigate the Thirring model any further, but refer the interested readers to Ref. [45], where a thorough study of this model with the gauge technique is given.

Before closing this section, we work out the explicit expressions for $S(x)$ in the covariant and the light-cone gauge. By using (14), we have

$$ig^2 \int \bar{d}^2 k \frac{1}{k} \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) e^{ikx} = i\pi \int \bar{d}^2 k \left(\frac{1}{k^2 - \mu^2 + i\epsilon} - \frac{1}{k^2 - a\mu^2 + i\epsilon} \right) e^{ikx}. \quad (36)$$

Applying the relation [73]

$$\int \bar{d}^2 k \frac{e^{ikx}}{k^2 - m^2 + i\epsilon} = \frac{1}{2\pi i} K_0(\overline{m\sqrt{-x^2 + i\epsilon}}), \quad (37)$$

we arrive at the explicit form of (36),

$$\text{i.e. } ig^2 \int \bar{d}^2 k \frac{1}{k} \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) e^{ikx} = \frac{1}{2} [K_0(\overline{\mu\sqrt{-x^2 + i\epsilon}}) - K_0(\overline{\mu/a\sqrt{-x^2 + i\epsilon}})] . \quad (38)$$

As x vanishes, (38) reduces to

$$ig^2 \int \bar{d}^2 k \frac{1}{k} \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) = \frac{1}{4} \ln a . \quad (39)$$

Substituting both (38) and (39) into (27) results in

$$S_A(x) = a^{-1/4} S_0(x) \cdot \exp \left\{ \frac{1}{2} [K_0(\overline{\mu\sqrt{-x^2 + i\epsilon}}) - K_0(\overline{\mu/a\sqrt{-x^2 + i\epsilon}})] \right\} . \quad (40)$$

Setting $a=1$ reproduces the result of Delbourgo and Shepherd [42].

To evaluate (31) we cast it into the form

$$S_S(x) = C(x_0) \cdot \exp \left[i g^2 \int d^2 k \frac{D_\mu^\mu(k)}{k^2} (e^{ikx} - e^{ikx_0}) \right], \quad (41)$$

with x_0 an arbitrary constant vector. Utilizing (37) we can easily work out that

$$S_S(x) = \frac{C'}{-x^2 + i\epsilon} \exp \left\{ -\frac{1}{2} [K_0(\mu\sqrt{-x^2 + i\epsilon}) + K_0(\mu\sqrt{a(-x^2 + i\epsilon)})] \right\}. \quad (42)$$

In the light-cone gauge, the vector meson propagator acquires the expression

$$D_{\mu\nu}(k) = \frac{-1}{k^2 - \mu^2 + i\epsilon} \left(\eta_{\mu\nu} - \frac{k_\mu n_\nu + n_\mu k_\nu}{n \cdot k} \right). \quad (43)$$

Applying (43) to (27) and (31), and carrying out similar calculations as we have done above, we obtain

$$S(x) = S_0(x) \exp \left\{ \frac{\not{x} \cdot \not{y}}{x \cdot n} [K_0(\mu\sqrt{-x^2 + i\epsilon}) + \ln(\mu\sqrt{-x^2 + i\epsilon})] \right\} + \text{Const.} \quad (44)$$

V.3 SCHWINGER MODEL AT FINITE TEMPERATURE

To close this chapter on the application of the gauge technique to Schwinger model we present the extension of the preceding results to finite temperature. In the functional formalism the temperature effect is incorporated into field theory by changing the time component integration path in the action and imposing the Schwinger-Martin-Kubo condition [87] on the Green functions. There various time integration paths are available, and with different paths one ends up with different formalisms of finite temperature field theories [88]. For example, the contour shown in Fig. [V-2] leads one to the so-called imaginary time formalism, which was first introduced by Matsubara [89] within a

canonical framework some three decades ago, and subsequently developed by Abrikosov and his co-workers [90] as well as other people [91, 92]. In this formalism, one completely loses information about time, and the fourth component of the momentum becomes discrete. On the other hand, if one adopts the contour shown in Fig. [V-3], one achieves a real-time formalism, where the degrees of freedom of the theory double up [93]; such a formalism is very similar to the canonical one (thermo-field-dynamics) developed by Umezawa, Matsumoto et al [94]. But it is an unfortunate fact that some formalisms disagree with one another in respect of perturbative calculations and this might be a sign of some inconsistency within these formalisms [95]. In this section we will not be concerned about formalisms, but stick to the imaginary-time formulation and extend the results of last section to this situation.

Following Ref. [91], we write the generating functional of the Schwinger model at finite temperature as

$$Z[\bar{J}, J, J_\mu] = N(\beta) \int [d\bar{\psi} d\psi dA_\mu] \exp \left\{ i \int_{-\infty}^{+\infty} dx \int_0^\beta d\tau \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\partial - g\beta) \psi + \mathcal{L}_\xi + J_\mu B^\mu + \bar{J}\psi + \bar{\psi} J \right] \right\}, \quad (45)$$

with \mathcal{L}_ξ the gauge fixing Lagrangian. β is the inverse temperature, defined by

$$\beta = \frac{1}{kT}, \quad (46)$$

k being the Boltzmann constant. The $B_\mu(x, \tau)$ field satisfies periodic boundary condition

$$B_\mu(x, \tau + i\beta) = B_\mu(x, \tau), \quad (47)$$

while $\psi(x, \tau)$ and $\bar{\psi}(x, \tau)$ are anti-periodic, that is

$$\psi(x, \tau + i\beta) = -\psi(x, \tau), \quad \bar{\psi}(x, \tau + i\beta) = -\bar{\psi}(x, \tau). \quad (48)$$

From (45) we can derive the Ward-Takahashi identities and the Dyson-Schwinger equations corresponding to (4), (5), (8) and (9), but with the following minor modifications

$$k_0 \rightarrow i\omega_n \quad , \quad \int \bar{d}^2 k \rightarrow \frac{i}{\beta} \sum_{k_0} \int \bar{d}k_1 \quad , \quad (49)$$

with $\omega_n = \frac{(2n+1)\pi}{\beta}$ for the spinor source and $\frac{2n\pi}{\beta}$ for the vector source.

Once we are clear about the above modifications, it becomes obvious that the argument for solving the Dyson-Schwinger equation (8) in the last section can also be applied to the finite temperature case, and we arrive at the solution

$$S_A(x) = S_0(x) \cdot \exp \left[- \frac{g^2}{\beta x \cdot \gamma} \sum_{k_0} \int \bar{d}k_1 \frac{1}{k} \gamma_\mu \frac{1}{k} \gamma_\nu D^{\mu\nu}(k) (e^{ikx} - 1)_{x \cdot \gamma} \right] \quad (50)$$

$$\text{and} \quad S_s(x) = C \cdot \exp \left[- \frac{g^2}{\beta} \sum_{k_0} \int \bar{d}k_1 \frac{D^\mu(k)}{k^2} e^{ikx} \right] .$$

Just as in the zero temperature case, (50) is valid for the Schwinger model in all gauges and also for the Thirring model. When set $a=0$ in (14) and (50) the $S_A(x)$ agrees with the result obtained by Stam and Visser [96].

Finally, we may point out that dynamical symmetry breaking (generation of vector meson mass) is not affected by temperature, in contrast to the Higgs mechanism, where symmetry can be restored when the temperature reaches a critical value [92]. Therefore at finite temperature equation (35) still holds.

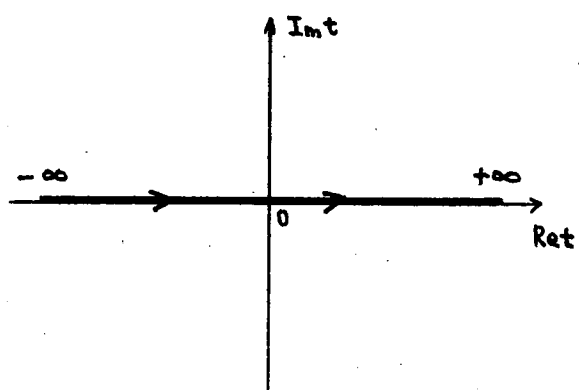


Fig.[V-1]. The time integration contour for ordinary field theory

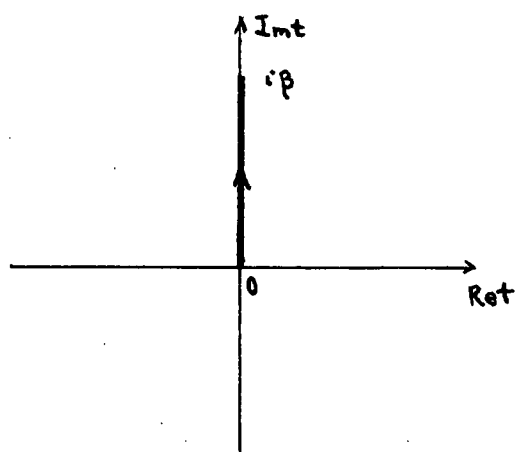


Fig.[V-2]. The contour for imaginary time formalism

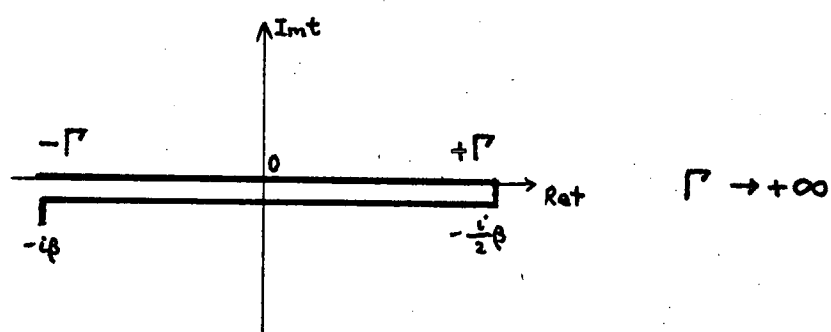


Fig.[V-3]. The contour for one of the real-time formalism

VI. GLUON PROPAGATOR AND TRANSVERSE VERTICES

As a non-perturbative method, the gauge technique is strikingly effective in reproducing the infrared structure of charged particle propagators in the context of quantum electrodynamics. This has tempted people to extend the method to infrared quantum chromodynamics so as to investigate the colour confinement problem. However, the results obtained by different people often disagree with one another and the problem is clouded by all manner of contradictory claims. The purpose of the present chapter is to shed some light on the situation.

VI.1 OUTLINE OF THE AXIAL GAUGE QUANTUM CHROMODYNAMICS

In considering the colour confinement problem, rather than studying the charged particle propagators, people usually choose to work in pure Yang-Mills theory (that is, with the quark-gluon interaction neglected) and concentrate on the gluon propagator. To avoid added complications caused by the fictitious particles in the gauge identities, the axial gauge ($n^2 \neq 0$) or sometimes the light-cone gauge ($n^2 = 0$) is used, in which case the gauge condition is

$$n^\mu A_\mu^a(x) = 0. \quad (1)$$

Where n^μ is an external constant vector, and the $A_\mu^a(x)$ is the gluon field. Let us denote the structure constants of the SU(3) colour group by C^{abc} . The action, incorporating an external source $J_\mu^a(x)$, reads

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^a(x) F^{\mu\nu a}(x) - J^{\mu a}(x) A_\mu^a(x) \right] \quad (2)$$

with $F_{\mu\nu}^a(x)$ the field strength tensor defined by

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g_0 C^{abc} A_\mu^b(x) A_\nu^c(x). \quad (3)$$

Where g_0 is the nurenormalized coupling constant, the subscripts and superscripts from the Greek and Latin alphabets are the space-time and colour indices respectively.

The generating functional with the gauge condition (1) is given by

$$Z[J] = \int [dA] \delta(n^\mu A_\mu^a(x)) \exp(iS) . \quad (4)$$

Where the δ -function should be understood in a functional sense.

By employing essentially the same functional operations as we did in Chapter II, we are led to the Dyson-Schwinger equation

$$\begin{aligned} \pi_{\mu\nu}(q) = & - (q^2 g_{\mu\nu} - q_\mu q_\nu) \\ & - i \frac{3}{2} g_0^2 \int d^4k \Gamma_{\mu\lambda\lambda}^{(0)}(q, -k, -k') \Delta^{\lambda\sigma}(k) \Delta^{\lambda'\sigma'}(k') \Gamma_{\sigma\sigma'\nu}(k, k', -q) \quad (5) \\ & + i \frac{3}{2} g_0^2 \int d^4k \Gamma_{\mu\lambda\sigma\nu}^{(0)}(q, k, -k, -q) \Delta^{\lambda\sigma}(k) + \text{four-gluon terms,} \end{aligned}$$

and the Slavnov-Taylor identity

$$\Gamma_{\sigma\sigma'\nu}(k, k', -q)(-q)^\nu = \pi_{\sigma\sigma'}(k) - \pi_{\sigma\sigma'}(k') . \quad (6)$$

where $\Delta_{\mu\nu}(q)$ denotes the gluon propagator, which is related to the 'inverse' propagator $\pi_{\mu\nu}(q)$ through

$$\pi_\mu^\lambda(q) \Delta_{\lambda\nu}(q) = g_{\mu\nu} - \frac{n_\mu q_\nu}{n \cdot q} , \quad (7)$$

and the Γ 's are the three- and four-point one-particle irreducible vertices respectively; their corresponding bare vertices are

$$\Gamma_{\mu\lambda\lambda}^{(0)}(q, -k, -k') = (q+k)_\lambda g_{\mu\lambda} + (k'-k)_\mu g_{\lambda\lambda} - (q+k')_\lambda g_{\mu\lambda} , \quad (8)$$

$$\Gamma_{\mu\lambda\sigma\nu}^{(0)}(q, k, -k, -q) = 2(g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\sigma} g_{\nu\lambda}) . \quad (9)$$

In the axial gauge, the bare gluon propagator assumes the expression,

$$\Delta_{\mu\nu}^{(0)}(q) = -\frac{1}{q^2} \left[g_{\mu\nu} - \frac{n_\mu q_\nu + n_\nu q_\mu}{n \cdot q} + \frac{n^2 q_\mu q_\nu}{(n \cdot q)^2} \right] . \quad (10)$$

As a reflection of the gauge condition (1), the propagator satisfies

$$n^\mu \Delta_{\mu\nu}(q) = \Delta_{\nu\mu}(q) n^\mu = 0 . \quad (11)$$

Here all the Green functions are unrenormalized; multiplicative renormalization can be easily carried out in the axial gauge (but not the light-cone gauge!) if we wish. For the sake of simplicity and clarity we have suppressed the colour indices. Spelling them out explicitly just once, we have

$$\Delta_{\mu\nu}^{ab}(q) = \delta^{ab} \Delta_{\mu\nu}(q), \pi_{\mu\nu}^{ab}(q) = \delta^{ab} \pi_{\mu\nu}(q), \Gamma_{\mu\nu\rho}^{abc} = C^{abc} \Gamma_{\mu\nu\rho}, \text{ etc. } (12)$$

For the remainder of this chapter, we will suppress the colour index in Green functions except where it is likely to cause confusion. Now we display equation (5) diagrammatically for latter use.

$$\pi_{\mu\nu}(q) = \pi_{\mu\nu}^{(0)}(q) + \text{[diagram of a gluon loop with a ghost loop]} + \text{[diagram of a gluon loop with a gluon loop]} + \text{[diagram of a gluon loop with a gluon loop]}$$

Fig.[VI-1]

VI.2 APPLICATIONS OF THE LOWEST ORDER GAUGE TECHNIQUE IN CHROMODYNAMICS

Although different researchers apply the lowest order gauge technique to chromodynamics in different ways, the essence of the argument is always the same: one uses an ansatz for the three gluon vertex, which satisfies the gauge identity, to truncate the Dyson-Schwinger equation for the gluon propagator and thereby arrive at an integral

equation for the gluon propagator (or the spectral functions of the propagator). The equation, then, is analysed in the infrared limit to uncover infrared behaviours of the gluon propagator, which is believed to be responsible for colour confinement by some people.

The first work in this area, by Delbourgo [56], relied on a spectral ansatz for the three-gluon connected Green function $\Delta\Delta\Gamma_{(3)}^\Delta$ which involved the same spectral functions as that of the gluon propagator.

Although the ansatz could reproduce the lowest order perturbation theory and possessed boson symmetry, it had the fatal deficiency that it was not multiplicatively renormalizable. (That ansatz led the author to an

infrared gluon propagator $\frac{1}{p^2 \ln(\frac{p^2}{\mu^2})}$ in the axial gauge. With the same

strategy Gardner [58] studied both three- and four-dimensional chromodynamics in the light-cone gauge and obtained two sets of solutions, one set of which behaved like $\frac{1}{p^3}$ and $\frac{1}{p^4}$ respectively for the infrared gluon propagator in three- and four-dimensions.)

Baker, Ball and Zachariasen [57] analysed the problem in a slightly different way. They designed an ansatz for the unrenormalized three-gluon vertex; then, with this ansatz they truncated the unrenormalized Dyson-Schwinger equation and carried out the necessary renormalizations. They claimed that numerical analysis of the resultant equation suggested a gauge invariant $\frac{1}{p^4}$ behaviour for the infrared gluon propagator - such behaviour being taken as an indication of colour confinement. Subsequently, people invested enormous amount of effort in the problem following these authors, despite West's claim [63] that in the axial gauge the gluon propagator could not be more singular than $\frac{1}{p^2}$ as required by the analyticity and positivity of the spectral functions of the gluon

propagator. Heck and Slim [97] in turn investigated the BBZ program in some detail, and found that the $\frac{1}{p^4}$ infrared structure would render the BBZ equation inconsistent unless the non-covariant argument $n \cdot p$ vanished. Their findings threw into question the BBZ result.

Instead of studying the gauge dependent gluon propagator directly, Cornwall [60] went much further and gave a more convincing treatment. By rearranging the Feynman graphs which contribute to physical amplitudes, he discovered that the gluon propagator could be replaced by an effective one which depended on the gauge (he worked in the light-cone gauge) only through the bare propagator. In this way he was able to circumvent the gauge dependence problem. He also found that a new Dyson-Schwinger equation existed relating the effective gluon propagator to a new vertex, and this new vertex obeyed a Ward identity with the effective propagator. Designing a spectral representation for the effective gluon propagator, he solved the Ward identity and obtained an approximation to the new vertex, which was then substituted into the Dyson-Schwinger equation. This time he arrived at a dynamically generated gluon mass of 500 ± 200 Mev and a 0^+ glueball mass of about twice this value.

With all these contradictory claims it is apparent that the argument is in urgent need of resolution. As our analysis in the next section is mainly related to the BBZ program, we present a detailed review of it in the following.

We follow the conventions of BBZ and work in Euclidean space. Let $\gamma = q^2 n^2 / (n \cdot q)^2$ stand for the gauge parameter. In the limit $q^2 \rightarrow 0$ (γ -fixed) the singular part of the gluon propagator was assumed by BBZ to have the same spin structure as the free propagator, that is

$$\Delta_{\mu\nu}(q) \rightarrow Z(q^2, \gamma) \Delta_{\mu\nu}^{(0)}(q) \quad , \quad (13)$$

with $Z(q^2, \gamma)$ an unknown scalar function of the variables q^2 and γ only.

Correspondingly there was the relationship

$$\pi_{\mu\nu}(q) \rightarrow - \frac{q^2}{Z(q^2, \gamma)} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \quad q^2 \rightarrow 0, \gamma\text{-fixed}. \quad (14)$$

Therefore, both $\pi_{\mu\nu}$ and $\Delta_{\mu\nu}$ are entirely determined by the $Z(q^2, \gamma)$ function. Based on (14), BBZ went to make an ansatz for the 3-gluon vertex which is consistent with the Slavnov-Taylor identity and free of kinematic singularities, v.z.

$$\begin{aligned} \Gamma_{\sigma_1 \sigma_2 \sigma_3 L}^{abc}(q_1, q_2, q_3) &= c^{abc} \{ g_{\sigma_1 \sigma_2} [Z^{-1}(q_1)(q_1)_{\sigma_3} - Z^{-1}(q_2)(q_2)_{\sigma_3}] \\ &\quad - \frac{Z^{-1}(q_1) - Z^{-1}(q_2)}{q_1^2 - q_2^2} [q_1 \cdot q_2 g_{\sigma_1 \sigma_2} - (q_2)_{\sigma_1} (q_1)_{\sigma_2}] (q_1 - q_2)_{\sigma_3} \} \\ &\quad + \text{cyclic permutations}. \end{aligned} \quad (15)$$

$\Gamma_{\sigma_1 \sigma_2 \sigma_3 L}^{abc}$ differs from the true three-gluon vertex by a purely transverse

part $\Gamma_{\sigma_1 \sigma_2 \sigma_3 T}^{abc}$

$$\text{i.e.} \quad \Gamma_{\sigma_1 \sigma_2 \sigma_3}^{abc}(q_1, q_2, q_3) = \Gamma_{\sigma_1 \sigma_2 \sigma_3 L}^{abc}(q_1, q_2, q_3) + \Gamma_{\sigma_1 \sigma_2 \sigma_3 T}^{abc}(q_1, q_2, q_3) \quad (16)$$

with

$$\Gamma_{\sigma_1 \sigma_2 \sigma_3 T}^{abc}(q_1, q_2, q_3)(q_3)^{\sigma_3} = 0. \quad (17)$$

$\Gamma_{\sigma_1 \sigma_2 \sigma_3 T}^{abc}$ vanishes identically when any one of the three momenta goes to zero. Based on the fundamental assumption that the contributions of

$\Gamma_{(3)L}$ to $\pi_{\mu\nu}$ dominated that of $\Gamma_{(3)T}$ in infrared, (15) was used by BBZ to replace the full $\Gamma_{(3)}$ in the Dyson-Schwinger equation (5). Contracting

the resultant equation with $\frac{n_\mu n_\nu}{n^2}$ they arrived at

$$\begin{aligned}
\frac{n_\mu \pi_{\mu\nu}(q) n_\nu}{n^2} &= - \frac{q^2}{Z(q)} \left(1 - \frac{1}{Y}\right) \\
&= q^2 \left(1 - \frac{1}{Y}\right) - g_0^2 \cdot \int dk \frac{n \cdot (k-k') n \cdot k'}{n^2} \Delta_{\gamma\sigma}^{(0)}(k) \Delta_{\gamma\sigma'}^{(0)}(k') \\
&\times \left\{ - \frac{Z(k)}{Z(q)} \cdot \frac{Z(k') - Z(q)}{k'^2 - q^2} \cdot (q+k')_{\sigma} q_{\sigma'} + \frac{Z(k) - Z(k')}{k^2 - k'^2} (k \cdot k' g_{\sigma\sigma'} - k'_\sigma k_{\sigma'}) \right. \\
&\quad \left. + Z(k) g_{\sigma\sigma'} \right\} \\
&+ g_0^2 \int dk \frac{Z(k)}{k^4} \left(2 + \frac{k^2 n^2}{(n \cdot k)^2}\right) , \tag{18}
\end{aligned}$$

where $\int dk$ is a shorthand for $N_c \int d^4 k$, with $N_c=3$ the number of colours.

Because (18) is an unrenormalized equation, divergences are present. To remove them, BBZ carried out 'mass' and 'charge' renormalizations.

The requirement, that the gluon be massless, entails that $\pi_{\mu\nu}(q)$ satisfies

$$\pi_{\mu\nu}(q) \rightarrow 0 , \quad \text{as } q \rightarrow 0 . \tag{19}$$

Therefore equation (18) should also meet this condition. However, (19) is only respected when dimensional regularization is adopted. Since (18) was to be analysed numerically the authors subtracted $\frac{n_\mu \pi_{\mu\nu}(0) n_\nu}{n^2}$ from (18) in order to guarantee (19). By letting $Z(k) = Z_1(k) + A/k^2$, with A a constant, they obtained

$$\begin{aligned}
\frac{n_\mu \pi_{\mu\nu}(0) n_\nu}{n^2} &= 2g_0^2 \int dk \frac{(n-k)^2}{n^2} \Delta_{\lambda\sigma}^{(0)}(k) \Delta_{\lambda\sigma'}^{(0)}(k) \left\{ Z_1(k) g_{\sigma\sigma'} \right. \\
&\quad \left. + \frac{\partial Z_1(k)}{\partial k^2} [-k^2 g_{\sigma\sigma'} + k_\sigma k_{\sigma'}] \right\} \\
&+ ig_0^2 \int dk Z_1(k) \left[2 + \frac{k^2 n^2}{(n \cdot k)^2} \right] \\
&- Ag_0^2 \int dk \frac{-2(k \cdot n)^2}{n^2} \cdot \Delta_{\lambda\sigma}^{(0)}(k) \Delta_{\lambda\sigma'}^{(0)}(k) \left\{ \frac{1}{k^2} (g_{\sigma\sigma'} - \frac{k_\sigma k_{\sigma'}}{k^2}) + \frac{1}{k^2} g_{\sigma\sigma'} \right\} . \tag{20}
\end{aligned}$$

After the subtraction was performed, they arrive at an equation for the $Z(q)$ function

$$\frac{1}{Z(q)} = 1 + g_0^2 \int dk K(k, q, n) Z(k) + \frac{g_0^2}{Z(q)} \int dk L(k, q, n) Z(k) Z(q-k) . \quad (21)$$

The kernels K and L can be read off from (18) and (20).

In order to carry out the 'charge' renormalization, BBZ introduced the renormalized coupling constant

$$g^2(M) = \frac{g_0^2 Z(M)}{1 + g_0^2 \int dk K(k, M, n) Z_R(k)} . \quad (22)$$

Where M was a fixed four-vector, and $Z_R(q)$ was defined by

$$Z(q) = Z(M) Z_R(q) . \quad (23)$$

Applying (22) and (23) to (21), the authors arrived at the following finite integral equation

$$\begin{aligned} \frac{1}{Z_R(q)} &= 1 + g^2(M) \int dk [K(k, q, n) - K(k, M, n)] \\ &+ \frac{g^2(M)}{Z_R(q)} \int dk [L(k, q, n) Z_R(q-k) - L(k, M, n) Z_R(M-k)] Z_R(k) , \end{aligned} \quad (24)$$

which they studied by trying different ansatz for $Z_R(q)$. They claimed that the only self-consistent solution was

$$Z_R(q) = AM^2/q^2 , \text{ as } q^2/M^2 \rightarrow 0 , \gamma\text{-fixed} , \quad (25)$$

with A a γ independent finite constant. Therefore the authors concluded that if (as they believed to be true but could not prove) the transverse vertex contribution was dominated by that of the longitudinal part in the infrared region, the gluon propagator behaved like

$$\Delta_{\mu\nu}(k) \rightarrow -\frac{M^2}{q} \left(g_{\mu\nu} - \frac{n_\mu q_\nu + n_\nu q_\mu}{n \cdot q} + \frac{n^2 q_\mu q_\nu}{(n \cdot q)^2} \right) , \quad (26)$$

and that although the gluon propagator in general depended on the gauge parameter γ , the constant A was gauge invariant in the infrared limit, therefore the propagator acquired physical significance and colour confinement was implied.

VI.3 GLUON PROPAGATOR AND TRANSVERSE VERTICES

Whether the axial gauge gluon propagator contains any physically meaningful information about colour confinement or not is still under debate, and we will not involve ourselves into this problem. Rather we consider the essential assumption made by the practitioners of the gauge technique: transverse contributions are negligible.

As we can see from the last section, the present problem bears much similarity to the one where the photon propagator is studied through equation (II-44). Because the vacuum polarization tensor

$$\pi_{\mu\nu}(k) = iZ_\psi e^2 \int d^4p \text{tr}[G_\mu(p, p-k)\gamma_\nu] \quad (27)$$

is known to be transversal to the photon momentum k , we can multiply both sides of (27) with the projection operator

$$T_{\mu\nu}(k) = g_{\mu\nu} - k_\mu k_\nu / k^2 \quad (28)$$

without altering it,

$$\text{i.e.} \quad \pi_{\mu\nu}(k) = iZ_\psi e^2 \int d^4p \text{tr}[G_\mu^{(T)}(p, p-k)\gamma_\nu] \quad (29)$$

$$\text{with} \quad G_\mu^{(T)}(p, p-k) = G_\nu(p, p-k)T_\mu^\nu(k) \quad (30)$$

Therefore the radiative corrections to the photon propagator consist solely of the contributions of the transverse part of G_μ . However, $G_\mu^{(T)}$ is completely irrelevant to the Ward-Takahashi identity

$$G_\mu(p', p)(p' - p)^\mu = S(p) - S(p') \quad (31)$$

We can not expect (31) by itself to enhance our knowledge about (29). Some other ingredients must be added, e.g., perturbation theory, pole dominance or whatever.

The same thing happens when the lowest order gauge technique is applied to explore the properties of the gluon propagator. We decompose the three-gluon vertex into two pieces, one of which is purely longitudinal, and the other transversal. They are defined by

$$\Gamma_{\sigma\sigma'\nu}^{(T)}(k,k',-q) = \Gamma_{\sigma\sigma'\mu}(k,k',-q)T^\mu_\nu(-q) , \quad (32a)$$

$$\Gamma_{\sigma\sigma'\nu}^{(L)}(k,k',-q) = \Gamma_{\sigma\sigma'\nu}(k,k',-q) - \Gamma_{\sigma\sigma'\nu}^{(T)}(k,k',-q) . \quad (32b)$$

This decomposition seems to introduce singularities into both $\Gamma_{(3)}^{(T)}$ and $\Gamma_{(5)}^{(L)}$ at $q^2=0$, but when we apply it to the Dyson-Schwinger equation, it turns out that the singularities do not matter. Substituting (32) into (5) and recalling the Slavnov-Taylor identity as well as (7), we arrive at

$$\begin{aligned} \pi_{\mu\nu}(q) = & -(q^2 g_{\mu\nu} - q_\mu q_\nu) \\ & - i \frac{3}{2} g_0^2 \int d^4k \Gamma_{\mu\lambda\lambda'}^{(0)}(q, -k, -k') \Delta^{\lambda\sigma}(k) \Delta^{\lambda'\sigma'}(k') \Gamma_{\sigma\sigma'\nu}^{(T)}(k, k', -q) \\ & - i \frac{3}{2} g_0^2 \int d^4k \Gamma_{\mu\lambda\lambda'}^{(0)}(q, -k, -k') [\Delta^{\lambda\lambda'}(k') - \Delta^{\lambda\lambda'}(k)] \left(-\frac{q_\nu}{q^2}\right) \\ & + i \frac{3}{2} g_0^2 \int d^4k \Gamma_{\mu\lambda\sigma\nu}^{(0)}(q, k, -k, -q) \Delta^{\lambda\sigma}(k) + \text{four-gluon terms} . \end{aligned} \quad (33)$$

By using the definition of $\Gamma_{(3)}^{(0)}$ and $\Gamma_{(4)}^{(0)}$ the third and fourth terms on the right-hand-side of (33) can be combined together, and the equation reduces to

$$\begin{aligned} \pi_{\mu\nu}(q) = & -(q^2 g_{\mu\nu} - q_\mu q_\nu) \\ & - i \frac{3}{2} g_0^2 \int d^4k \Gamma_{\mu\lambda\lambda'}^{(0)}(q, -k, -k') \Delta^{\lambda\sigma}(k) \Delta^{\lambda'\sigma'}(k') \Gamma_{\sigma\sigma'\nu}^{(T)}(k, k', -q) \\ & - i 3 g_0^2 \int d^4k [g_{\nu}^\sigma - q_\nu q^\sigma / q^2] \cdot [\Delta_{\mu\sigma}(k) - g_{\mu\sigma} \Delta_\lambda^\lambda(k)] + \text{four-gluon terms} . \end{aligned} \quad (34)$$

Basically what we have done is to absorb the $\Gamma_{(3)}^{(L)}$ part of the original equation into a tadpole term.

The four-gluon terms, being hard to analyse, are eliminated by multiplying (34) with n^+ , thereby, reducing equation (34) to the once-subtracted form

$$\begin{aligned} n^\mu \pi_{\mu\nu}(q) = & -(q^2 n_\nu - n \cdot q q_\nu) \\ & + i \frac{3}{2} g_0^2 \int d^4 k \, n \cdot (k-k') \Delta_\lambda^\sigma(k) \Delta^{\lambda\sigma'}(k') \Gamma_{\sigma\sigma'\nu}^{(T)}(k, k', -q) \\ & + i 3 g_0^2 \int d^4 k \Delta_\lambda^\lambda(k) [n_\nu - q_\nu n \cdot q / q^2] . \end{aligned} \quad (35)$$

Here the third term deserves special consideration. If, as people often assume, it vanishes identically under dimensional regularization, then $\Gamma_{(3)}^{(T)}$ gives the total radiative corrections to the 'inverse' propagator. However, we can not really rule out the possibility that $\int d^4 k \Delta_\lambda^\lambda(k)$ is regularized to a q independent constant. Even in this case, the third term of (35) can not possibly dominate the second one, because it must be cancelled away by certain contributions from $\Gamma_{(3)}^{(T)}$, in order for the gluon to remain massless, that is for $\pi_{\mu\nu}(q)$ to satisfy (19). Specifically we may decompose $\Gamma_{(3)}^{(T)}$ into two parts,

$$\Gamma_{\sigma\sigma'\nu}^{(T)}(k, k', -q) = \Gamma_{\sigma\sigma'\nu}^{(T')} (k, k', -q) - \Gamma_{\sigma\sigma'\nu}^{(T'')} (k, k', -q) . \quad (36)$$

with $\Gamma_{\sigma\sigma'\nu}^{(T'')}$ defined by

$$n \cdot (k-k') \Gamma_{\sigma\sigma'\nu}^{(T'')} (k, k', -q) = [\pi_{\sigma\sigma'}(k) + \pi_{\sigma\sigma'}(k')] [n_\nu - \frac{q_\nu n \cdot q}{q^2}] \quad (37)$$

+ The last diagram of Fig. [VI-1] is $\int \Gamma_{(4)}^{(0)} \Delta \Delta \Delta \Gamma_{(4)}$. Because the bare four-gluon vertex $\Gamma_{(4)}^{(0)}$ consists of combinations of the metric tensor $g_{\mu\nu}$ only, it becomes obvious, by recalling (11), that $\int n \cdot \Gamma_{(4)}^{(0)} \Delta \Delta \Delta \Gamma_{(4)} \equiv 0$. The same argument applies to the other four-gluon term.

and with the tadpole term removed by the contributions of $\Gamma_{(3)}^{(T'')}$. In this way (35) is simplified to

$$n^\mu \pi_{\mu\nu}(q) = -(q^2 n_\nu - q_\nu n \cdot q) + i \frac{3}{2} g_0^2 \int \bar{d}^4 k n \cdot (k-k') \Delta_\lambda^\sigma(k) \Delta^{\lambda\sigma'}(k') \Gamma_{\sigma\sigma'\nu}^{(T')}(k, k', -q). \quad (38)$$

Like $\Gamma_{\sigma\sigma'\nu}^{(T)}(k, k', -q)$ itself, $\Gamma_{\sigma\sigma'\nu}^{(T')}(k, k', -q)$ is purely transversal to q through the subscript ν ; consequently it plays no role in the gauge identity.

The point to be stressed is that, contrary to common belief, the purely longitudinal three-gluon vertex does not contribute to the inverse propagator; rather the radiative corrections to it lie solely in a transverse part of the vertex, which is completely irrelevant at the level of the lowest Slavnov-Taylor identity. Thus in the axial gauge the gauge identity (6) by itself contains no useful information for determining the gluon propagator through the Dyson-Schwinger equation.

Up to now we have made neither assumptions nor approximations; (38) is an exact equation and is true in any momentum region. Multiplying it with $\frac{n^\nu}{n^2}$ results in an equation more general than (18). Therefore the radiative corrections to (18) consist of the contributions of a transverse piece of $\Gamma_{(3)L}$ defined by (15); furthermore, this transverse part is inherently arbitrary, although (15) satisfies another requirement that it be free of any kinematic singularity. Jennings and Woloshyn [98] argued that this was a sufficient condition to guarantee that the $\Gamma_{(3)T}$ defined in (16) vanished as any one of the three momenta went to zero, and particularly, $\Gamma_{(3)T}$ was at least a power of q_3 higher than $\Gamma_{(3)L}$ when q_3 was vanishing. However this does not guarantee that the contributions of the transverse piece contained in $\Gamma_{(3)L}$ to $\pi_{\mu\nu}(q_3)$ dominate that of

$\Gamma_{(3)T}$ in the vanishing q_3 limit, because the $\pi_{\mu\nu}(q_3)$ itself is at least of the order $O(q_3^2)$.

Other researchers have also explored (5) in slightly different ways [56] [58]. Instead of making ansatz for $\Gamma_{(3)}$ in accordance with the Slavnov-Taylor identity (6), they have tried to obtain an expression for $(\Delta\Gamma\Delta)_{\mu\nu\rho}(k,k',-q)$ which satisfies

$$(\Delta\Gamma)_{\mu\nu\rho}(k,k',-q) \cdot (-q)^\rho = \Delta_{\mu\nu}(k') - \Delta_{\mu\nu}(k) . \quad (39)$$

Of course this does not intrinsically change the problem, for although they are able to include a transverse piece in $\Gamma_{(3)}$, it is non-unique unless one introduces further criteria. Hence the resultant equation, obtained by applying their ansatz to (5), is still somewhat arbitrary (although, to be fair, they do attempt to incorporate the lowest order perturbation theory in their work).

These remarks partially apply to Cornwall's method [60]. His effective gluon propagator obeys a Dyson-Schwinger equation, of which the first three diagrams are exactly those in Fig.[VI-1] but with the gluon propagator and three-gluon vertex replaced by the improved gauge invariant quantities. The new gauge identity assumes the form

$$k_1^\alpha \hat{\Gamma}_{\alpha\beta\gamma}(k_1, k_2, k_3) = \hat{\Delta}_{\beta\gamma}^{-1}(k_2) - \hat{\Delta}_{\beta\gamma}^{-1}(k_3) , \quad (40)$$

with $\hat{\Gamma}_{(3)}$ and $\hat{\Delta}$ the effective vertex and propagator respectively. With the $\hat{\Delta}$ designed so that its gauge dependence resides solely in the bare propagator, the vacuum polarization tensor can be written as

$$\hat{\pi}_{\mu\nu}(k) = -(k^2 g_{\mu\nu} - k_\mu k_\nu) \hat{\pi}(k^2) , \quad (41)$$

and is purely transversal to k . Again $\hat{\pi}_{\mu\nu}$ can only consist of

contributions of transverse pieces of $\hat{\Gamma}_{(3)}$ and such pieces could not possibly be determined by the identity (40). But in constructing the ansatz for $(\hat{\Delta}(k_1)\hat{\Delta}(k_2)\hat{\Gamma}(k_1,k_2,k))_{\mu\nu\rho}$, Cornwall required the discarded transverse part to vanish when k went to zero. Because the gluon acquires a mass, the above requirement does guarantee that the purely transverse piece of Cornwall's ansatz gives $\hat{\pi}_{\mu\nu}(k)$ the correct $O(1)$ term; however the $O(k^2)$ and higher order terms are still arbitrary and we would therefore claim that the author can not predict the gluon mass quantitatively very precisely with the present method - his treatment is only qualitatively meaningful.

In summary we wish to point out that the work done in this area so far is contradictory and inconclusive; the lowest order gauge technique is in principle irrelevant to massless infrared chromodynamics. Even Cornwall's approach which is on firmer ground can only lead to a qualitative result, namely, the dynamical generation of a gluon mass. However when we ascend to the next level of this method by properly determining the transverse vertices it should be possible to uncover certain interesting properties of the gluon propagator. This research remains to be done.

VII. CONCLUSION

The present chapter summarizes the material presented in this thesis and highlights possible applications of the gauge technique in future research.

VII.1 SUMMARY

The main body of this thesis commenced at Chapter II, where we outlined the ingredients of the gauge technique. Using functional method we derived the first few Dyson-Schwinger equations and a general formula for the gauge identities for spinor and scalar electrodynamics, and explicitly demonstrated the consistency between these equations and identities. We also briefly presented the features of the Källén-Wightman-Lehmann spectral representations, and in the final section, sketched out the corresponding formalism in axial gauges.

Chapter III was devoted to the application of the gauge technique to spinor electrodynamics. Following a detailed review of the lowest order gauge technique, we developed a non-perturbative method of introducing transverse vertices, representing the next phase of evolution of the gauge technique. The refined gauge technique equation for the propagator was renormalized to a finite integral equation, and in the infrared limit we were still able to recover the standard infrared behaviour. The introduction of the transverse vertex provided a new ultra-violet behaviour and also improved the gauge property of the vacuum polarization tensor, rendering it gauge independent to order e^4 .

In Chapter IV we began by reviewing the lowest order gauge technique in scalar electrodynamics and Parker's approach [54] towards introducing transverse vertices; then we extended his work from the Feynman gauge to an arbitrary gauge specified by the gauge parameter a .

Our work eventually led to a finite integral equation for the spectral function, which unambiguously produced the standard infrared solution, in contrast to Parker's equation (which contained an arbitrary constant that could only be fixed by requiring the standard infrared behaviour for the spectral function). Solving this equation in the ultra-violet region we were also able to determine the correction to the ultra-violet behaviour of the meson propagator induced by the transverse vertex. The refined vertex function was also applied to analyse the vacuum polarization tensor, and we verified that to order e^4 it was exactly gauge independent. Finally we turned to the axial gauge and obtained the spectral function valid at all momenta to lowest order in the technique.

Chapter V was a self-contained section on two-dimensional models. Utilizing the exact vertex function obtained by Delbourgo and Thompson [44] (who solved the axial and vector gauge identities), we examined the Dyson-Schwinger equation for the spinor meson propagator of the massless Schwinger model, without referring to the spectral representation of this propagator. We were able to get an exact solution applicable to any gauge (by the same stroke this provided the exact solution to the Thirring model), and a corresponding massive vector meson propagator. Finally we generalized the solution to two-dimensional models at finite temperature using the imaginary-time formalism.

We discussed quantum chromodynamics in Chapter VI. After a brief review of the work done in this area so far, we made a detailed examination of the BBZ program [57]. By manipulating the Dyson-Schwinger equation for the gluon propagator in a similar way to BBZ, we arrived at the conclusion that only the transverse piece of the three-gluon vertex is relevant for the gluon self-energy. Therefore the Slavnov-Taylor

identity by itself contains no useful information for determining the gluon propagator through the Dyson-Schwinger equation; the lowest order gauge technique is in principle helpless when it comes to massless chromodynamics. However in Cornwall's approach [60] the technique is more successful and gives a qualitative result that the gluon acquires a dynamical mass, even if the quantitative answers are not entirely convincing (since they do depend on uncalculated transverse parts).

VII.2 OUTLOOK

Because of the algebraic complexity, the refined gauge technique equations in both spinor and scalar electrodynamics are analytically intractable in the intermediate momentum ($p \sim 3m$) region, but it should certainly be possible to undertake a numerical analysis of these equations. Once a numerical spectral function for the electron propagator is worked out, we can study its gauge properties and find out whether our transverse vertex has completely restored gauge covariance to the gauge technique - the restoration is of course exact to order e^4 . The transverse vertex can also be applied to calculate the form factors of an electron in a non-perturbative way. Perhaps in this way it will be possible to uncover some new features of the form factors, which can not be revealed by perturbation theory, such as the asymptotic behaviour in the momentum transfer.

Finally we want to point out that the gauge technique may be generalized to four-dimensional theories at finite temperature [99], but certain modifications of the conventional method may be necessary. Although spectral representations for two-point Green functions exist in finite temperature field theories, they are not readily amenable to the gauge technique. Therefore, we may have to follow the approach we have

established for two dimensional models in the final section of Chapter V, where we abandoned the spectral representation and solved the Dyson-Schwinger equation as a whole rather than studying the discontinuity of the propagator. Here the first problem confronting us is how to solve the gauge identity simply to obtain a sensible vertex function.

APPENDIX A. THE EVALUATION OF $\Sigma(p, W)$

The mass operator $\Sigma(p, W)$ consists of four pieces

$$\Sigma(p, W) = \Sigma^{(0)}(p, W) + \Sigma_Y(p, W) + L(p, W) + I(p, W) . \quad (1)$$

$\Sigma^{(0)}$ is the $O(e^2)$ contribution, $\Sigma_Y(p, W)$ is vacuum polarization contribution and $L(p, W)$, $I(p, W)$ are the contributions of the transverse vertex.

Because only the discontinuity of (1) and the real part of $\Sigma^{(0)}$ enter the Delbourgo-West equation, we will merely evaluate these two quantities explicitly. To handle the ultra-violet divergences, we adopt a comparatively old fashioned regularization scheme [70], that is, to introduce an L^{-2} dimensional quantity Λ^2 to cut off the divergent integrals at high momenta. In order for our calculations to be consistent, we cut off all integrals at the same momentum, and therefore, always maintain the same Λ^2 .

Consider $\Sigma^{(0)}(p, W)$ first. Its definition is given in (III-7).

Here we rearrange the terms into

$$\Sigma^{(0)}(p, W) = -ie^2 \int d^4 \ell \left\{ \frac{-2\ell}{[(p-\ell)^2 - W^2]\ell^2} + \frac{(1+a)\not{p} - (3+a)W}{[(p-\ell)^2 - W^2]\ell^2} + (1-a) \frac{(p^2 - W^2)\ell}{[(p-\ell)^2 - W^2]\ell^4} \right\} . \quad (2)$$

Then we evaluate the following integrals

$$\begin{aligned} ie^2 \int d^4 \ell \frac{1}{[(p-\ell)^2 - W^2]\ell^2} &= -\left(\frac{e}{4\pi}\right)^2 \left[\ln \frac{\Lambda^2}{W^2} + 1 + \frac{p^2 - W^2}{p^2} \ln \left| \frac{W^2}{p^2 - W^2} \right| \right] \\ &\quad - i\pi \left(\frac{e}{4\pi}\right)^2 \cdot \frac{p^2 - W^2}{p^2} \theta(p^2 - W^2) , \\ ie^2 \int d^4 \ell \frac{\ell}{[(p-\ell)^2 - W^2]\ell^2 \not{p}} &= -\frac{1}{2} \left(\frac{e}{4\pi}\right)^2 \left[\ln \frac{\Lambda^2}{W^2} + \frac{p^2 - W^2}{p^2} + \left(\frac{p^2 - W^2}{p^2}\right) \ln \left| \frac{W^2}{p^2 - W^2} \right| \right] \\ &\quad - \frac{i}{2} \pi \left(\frac{e}{4\pi}\right)^2 \cdot \left(\frac{p^2 - W^2}{p^2}\right) \theta(p^2 - W^2) , \\ ie^2 \int d^4 \ell \frac{\not{p} \cdot \ell}{[(p-\ell)^2 - W^2]\ell^4} &= -\left(\frac{e}{4\pi}\right)^2 \cdot \left[1 - \frac{W^2}{p^2} \ln \left| \frac{W^2}{p^2 - W^2} \right| \right] + i\pi \left(\frac{e}{4\pi}\right)^2 \cdot \frac{W^2}{p^2} \theta(p^2 - W^2) . \end{aligned} \quad (3)$$

Substituting (3) into (2), we obtain

$$\text{Im}\Sigma^{(0)}(p, W) = \pi \left(\frac{e}{4\pi}\right)^2 \frac{p^2 - W^2}{p^2} \left[a \not{p} \frac{p^2 + W^2}{p^2} - (3+a) \cdot W \right] \theta(p^2 - W^2) \quad (4)$$

and

$$\begin{aligned} \text{Re}\Sigma^{(0)}(p, W) = & \left(\frac{e}{4\pi}\right)^2 \cdot \{-W(a+3) \left[\ln \frac{\Lambda^2}{W^2} + 1 + \frac{p^2 - W^2}{p^2} \ln \left| \frac{W^2}{p^2 - W^2} \right| \right] \right. \\ & \left. + a \not{p} \left[\ln \frac{\Lambda^2}{W^2} + 1 - \frac{p^2 - W^2}{p^2} + \frac{p^4 - W^4}{p^4} \ln \left| \frac{W^2}{p^2 - W^2} \right| \right] + \not{p} \} \cdot \quad (5) \end{aligned}$$

(4) was worked out before [35]. In the $\not{p}=W$ limit (5) reduces to

$$\text{Re}\Sigma^{(0)}(W, W) = \left(\frac{e}{4\pi}\right)^2 \cdot \{-3W \left[\ln \frac{\Lambda^2}{W^2} + 1 \right] + W\} \quad (6)$$

To evaluate I and L, we note that the photon dressing can be neglected and the longitudinal part of the photon propagator $D^{\mu\rho}(\ell)$ does not contribute. Therefore we can redefine I and L as

$$\begin{aligned} I(p, W) = & ie^2 \int d^4\ell \{ \Sigma^{(0)}(p, W) \frac{\not{p}\gamma_\mu + \gamma_\mu(\not{p}-\not{\ell})}{p^2 - (p-\ell)^2} - \\ & - \frac{\not{p}\gamma_\mu + \gamma_\mu(\not{p}-\not{\ell})}{p^2 - (p-\ell)^2} \Sigma^{(0)}(p-\ell, W) \} \frac{D_F^{(0)\mu\rho}(\ell)}{(\not{p}-\not{\ell})-W} \quad (7) \end{aligned}$$

$$L(p, W) = e^4 \int d^4\ell d^4k \gamma_\nu \frac{1}{\not{p}-\not{k}-W} \gamma_\mu \frac{1}{\not{p}-\not{k}-\not{\ell}-W} \gamma_\lambda \frac{1}{\not{p}-\not{\ell}-W} \gamma_\rho D^{(0)\nu\lambda}(k) D_F^{(0)\mu\rho}(\ell) \quad (8)$$

with $D_F^{(0)\mu\rho}(\ell)$ the bare photon propagator in the Feynman gauge

$$D_F^{(0)\mu\rho}(\ell) = -\frac{g_{\mu\nu}}{\ell^2} \quad (9)$$

(7) is easier to evaluate than (8), so we consider it first. Utilizing the dispersion relation⁺

⁺ (10) can be proven by defining

$$\Sigma^{(0)}(p, W) = A^{(0)}(p^2, W^2) W + B^{(0)}(p^2, W^2) \not{p}$$

and using the usual once-subtracted dispersion relation to $A^{(0)}$ and $B^{(0)}$.

$$\Sigma^{(0)}(p, W) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{W' - p - i\epsilon(W')0^+}, \quad (10)$$

we can cast (7) into the form

$$I(p, W) = ie^2 \int d^4 \ell \frac{1}{\pi} \int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{p - W'} \gamma_\mu \frac{1}{p - \ell - W} \frac{1}{p - \ell - W'} \gamma_\rho D_F^{(0)\mu\rho}(\ell). \quad (11)$$

By recalling the definition of $\Sigma^{(0)}$, it is almost trivial to manipulate (11) into

$$I(p, W) = - \frac{1}{\pi} \int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{(W' - W)(p - W')} [\Sigma_F^{(0)}(p, W') - \Sigma_F^{(0)}(p, W)]. \quad (12)$$

Where $\Sigma_F^{(0)}$ denotes the $\Sigma^{(0)}$ in the Feynman gauge, $a=1$, and the $i\epsilon(W')0^+$ description is understood. Taking the imaginary part of (12) we arrive at

$$\begin{aligned} \text{Im}I(p, W) = & - \frac{1}{\pi} \int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{(W' - W)(p - W')} \text{Im}[\Sigma_F^{(0)}(p, W') - \Sigma_F^{(0)}(p, W)] \\ & + \int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{(W' - W)} (p + W') \text{Re}[\Sigma_F^{(0)}(p, W') - \Sigma_F^{(0)}(p, W)] \delta(p^2 - W'^2). \end{aligned} \quad (13)$$

Picking up the various integrals entering (13) and working them out separately, we obtain

$$\begin{aligned} \int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{(W' - W)(p - W')} &= \pi \left[\frac{p+W}{p^2 - W^2} [A^{(0)}(W^2, W^2)W + pB^{(0)}(W, W) - \text{Re}\Sigma^{(0)}(p, W)] \right. \\ &\quad \left. - B^{(0)}(W^2, W^2) \right] \theta(p^2 - W^2), \\ \int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)}{(W' - W)(p - W')} \text{Im}\Sigma_F^{(0)}(p, W') &= \pi^2 \left(\frac{e}{4\pi} \right)^4 \frac{p^2 - W^2}{p^2} \{ -(a+3) \left[\left(\frac{p^2}{p^2 - W^2} \ln \frac{p^2}{W^2} - 3 \right) W \right. \right. \\ &\quad \left. \left. + p \frac{W^2}{p^2} \left(1 - \frac{3p^2}{p^2 - W^2} \ln \frac{p^2}{W^2} \right) \right] + a \left[(-2 + \frac{p^2 - 3W^2}{p^2 - W^2} \ln \frac{p^2}{W^2}) W \right. \right. \\ &\quad \left. \left. + p \left(\frac{3W^2 - 5p^2}{2p^2} - 3 \frac{W^2}{p^2 - W^2} \ln \frac{p^2}{W^2} \right) \right] \right\} \theta(p^2 - W^2), \end{aligned} \quad (14)$$

$$\int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)(p+W')}{W'-W} \delta(p^2-W'^2) \frac{W'}{W} = \frac{p+W}{p^2-W^2} \text{Im}\Sigma^{(0)}(p, W) \frac{p}{W} ,$$

$$\int dW' \frac{\epsilon(W') \text{Im}\Sigma^{(0)}(W', W)(p+W')}{W'-W} \delta(p^2-W'^2) = \frac{p+W}{p^2-W^2} \text{Im}\Sigma^{(0)}(p, W) .$$

Inserting these integrals in (13) and simplifying the expression as much as we can, we still get the complicated result

$$\begin{aligned} \text{Im}I(p, W) = & -B^{(0)}(W^2, W^2) \text{Im}\Sigma_F^{(0)}(p, W) + A_F^{(0)}(p^2, p^2) \text{Im}\Sigma^{(0)}(p, W) \\ & + \left(\frac{e}{4\pi}\right)^2 \frac{p+W}{p^2} \text{Im}\Sigma^{(0)}(p, W) \left\{ \frac{(4W-p)p^2}{p^2-W^2} \ln \frac{p^2}{W^2} + 2(4W-p) \frac{p^2+W^2}{p^2} \ln \frac{W^2}{p^2-W^2} + p \right\} \\ & - \pi \left(\frac{e}{4\pi}\right)^4 \frac{p^2-W^2}{p^2} \left\{ -(a+3) \left[\left(\frac{4W^2}{p^2-W^2} \ln \frac{p^2}{W^2} - 3\right) \cdot W + p \frac{W^2}{p^2} \right] \right. \\ & \left. + a \left[\left(\frac{p^2-W^2}{p^2} + 4 \frac{p^2}{p^2-W^2} \ln \frac{p^2}{W^2}\right) W + p \left(\frac{9W^2-7p^2}{2p^2} + \frac{4W^4-p^4}{p^2(p^2-W^2)} \ln \frac{p^2}{W^2} \right) \right] \right\} \quad (15) \\ & \times \theta(p^2-W^2), \end{aligned}$$

where

$$\begin{aligned} B^{(0)}(W^2, W^2) &= \left(\frac{e}{4\pi}\right)^2 \left[a \left(\ln \frac{\Lambda^2}{W^2} + 1 \right) + 1 \right] \\ A_F^{(0)}(p^2, p^2) &= -4 \left(\frac{e}{4\pi}\right)^2 \left[\ln \frac{\Lambda^2}{p^2} + 1 \right] . \end{aligned} \quad (16)$$

Let us start evaluating $L(p, W)$. First, we decompose it into a gauge dependent part $L_g(p, W)$ and gauge independent part $L_i(p, W)$,

$$\text{i.e.} \quad L(p, W) = L_g(p, W) + L_i(p, W) , \quad (17)$$

$$\text{with} \quad L_g(p, W) = -(1-a)e^4 \int \bar{d}^4 \ell \int \bar{d}^4 k \, k \frac{1}{(p-k)-W} \gamma_\mu \frac{1}{(p-\ell-k)-W} k \frac{1}{(p-\ell)-W} \gamma^\mu \frac{1}{k^2 \ell^2} \quad (18)$$

$$\text{and} \quad L_i(p, W) = e^4 \int \bar{d}^4 \ell \int \bar{d}^4 k \, \gamma_\nu \frac{1}{p-k-W} \gamma_\mu \frac{1}{p-\ell-k-W} \gamma^\nu \frac{1}{p-k-W} \gamma^\mu \frac{1}{k^2 \ell^2} . \quad (19)$$

Recalling the definition of $\Sigma_F^{(0)}$, we can express (18) as

$$L_g(p, W) = i(1-a)e^2 \int \bar{d}^4 k \, k \frac{1}{(\not{p}-\not{k})-W} \cdot \frac{1}{k^4} [\Sigma_F^{(0)}(p-k, W) - \Sigma_F^{(0)}(p, W)] \quad (20)$$

A further decomposition of (20), that is, $L_g = L_{g_1} + L_{g_2}$, leads us to

$$L_{g_1}(p, W) = -i(1-a)e^2 \int \bar{d}^4 k \, k \frac{1}{(\not{p}-\not{k})-W} \cdot \frac{1}{k^4} \Sigma_F^{(0)}(p, W) \quad (21)$$

$$L_{g_2}(p, W) = +i(1-a)e^2 \int \bar{d}^4 k \, k \frac{1}{(\not{p}-\not{k})-W} \cdot \frac{1}{k^4} \Sigma_F^{(0)}(p-k, W) \quad (22)$$

With the aid of the integrals (3), the absorptive part of (21) can be worked out without too much difficulty

$$\begin{aligned} \text{Im} L_{g_1}(p, W) = & - \left(\frac{e}{4\pi}\right)^2 (1-a) \text{Im} \Sigma_F^{(0)}(p, W) \left\{ \ln \frac{\Lambda^2}{W^2} + \ln \frac{W^2}{p^2 - W^2} - \frac{W}{p} \left[1 - \frac{W^2}{p^2} \ln \frac{W^2}{p^2 - W^2} \right] \right\} \\ & - \pi \left(\frac{e}{4\pi}\right)^2 (1-a) \text{Re} \Sigma_F^{(0)}(p, W) \left[1 + \frac{W}{p} \frac{W^2}{p^2} \right] \theta(p^2 - W^2) \quad (23) \end{aligned}$$

L_{g_2} is not much more difficult than L_{g_1} . By inserting (10) into (22), we arrive at the following expression

$$L_{g_2}(p, W) = \frac{1}{\pi} \int dW' \frac{\epsilon(W') \text{Im} \Sigma^{(0)}(W', W)}{W' - W} (1-a) i e^2 \int \bar{d}^4 k \, k \frac{1}{k^4} \left[\frac{1}{\not{p}-\not{k}-W} - \frac{1}{\not{p}-\not{k}-W'} \right] \quad (24)$$

Once again the integrals of (3) are of assistance, and the absorptive part of (24) is worked out to be

$$\begin{aligned} \text{Im} L_{g_2}(p, W) = & (1-a) \left(\frac{e}{4\pi}\right)^2 \cdot \int dW' \frac{\epsilon(W') \text{Im} \Sigma^{(0)}(W', W)}{W' - W} \left[\left(1 + \frac{W'^3}{p^4}\right) \theta(p^2 - W'^2) \right. \\ & \left. - \left(1 + \frac{W'^3}{p^4}\right) \theta(p^2 - W^2) \right] \quad (25) \end{aligned}$$

Explicit evaluation of the integrals in (25) leads us to

$$\begin{aligned} \text{Im} L_{g_2}(p, W) = & \pi \left(\frac{e}{4\pi}\right)^2 \cdot (1-a) \left(1 + \frac{p W^3}{p^4}\right) \Sigma_F^{(0)}(W, W) \theta(p^2 - W^2) \\ & - \pi \left(\frac{e}{4\pi}\right)^4 (1-a) \cdot \frac{p^2 - W^2}{p^2} \left\{ W \left[1 - 3 \frac{p^2}{p^2 - W^2} \ln \frac{p^2}{W^2} \right] + \frac{p^2 - 5W^2}{2p} \right\} \theta(p^2 - W^2) \quad (26) \end{aligned}$$

For the sake of clarity, we combine (23) and (26) together and spell out the explicit form of $\text{Im}L_g(p,W)$,

$$\begin{aligned}
 \text{Im}L_g(p,W) = & -\pi\left(\frac{e}{4\pi}\right)^4 \cdot (1-a) \cdot \frac{p^2-W^2}{p^2} [-4W + \not{p} \frac{p^2+W^2}{p^2}] \cdot \left[\ln \frac{\Lambda^2}{W^2} + 1\right] \theta(p^2-W^2) \\
 & + \pi\left(\frac{e}{4\pi}\right)^4 \cdot (1-a) \cdot \frac{p^2-W^2}{p^2} [W - \not{p} \frac{p^2+W^2}{p^2}] \cdot \left[\ln \frac{\Lambda^2}{W^2} + 1\right] \theta(p^2-W^2) \\
 & - \pi\left(\frac{e}{4\pi}\right)^4 \cdot (1-a) \cdot \frac{p^2-W^2}{p^2} \left\{ W \left[\frac{3p^2-2W^2}{p^2} + 2 \frac{W^4+p^2W^2-4p^4}{p^4} \ln \frac{W^2}{p^2-W^2} \right. \right. \\
 & \left. \left. - 3 \frac{W^2}{p^2-W^2} \ln \frac{p^2}{W^2} \right] + \not{p} \left[\frac{3W^2-p^2}{2p^2} + 2 \frac{p^2W^2+p^4-4W^4}{p^4} \ln \frac{W^2}{p^2-W^2} \right] \right\} \\
 & \times \theta(p^2-W^2) .
 \end{aligned} \tag{27}$$

Diagrammatically $L_i(p,W)$ can be expressed as

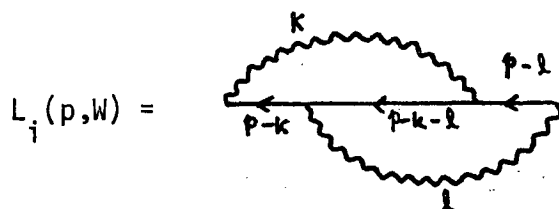


Fig. [A-1]

By applying the Cutkosky-Nakanishi [82] cutting rules to Fig. [A-1] the imaginary part comes by letting various combinations of propagators go on-shell. Adopting the notation that a small bar through a propagator takes us on-shell, we have

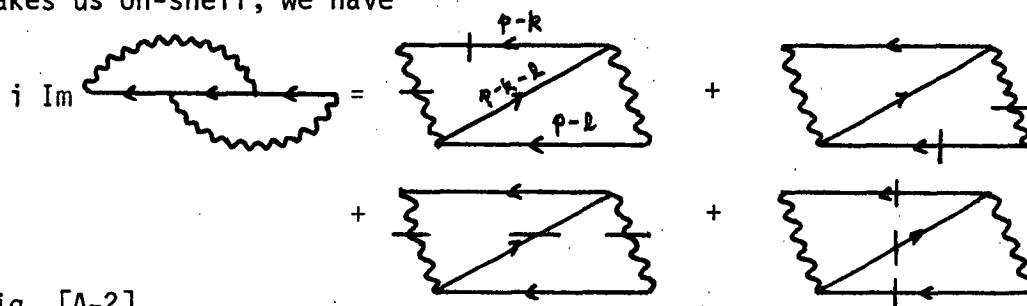


Fig. [A-2].

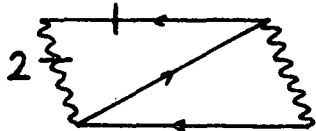
To make our calculations simpler, we disentangle the γ -algebra. Noting the relationship

$$L_i(p,W) = \frac{1}{4} \text{tr}(L_i(p,W)) + \frac{1}{4\not{p}} \text{tr}(\not{p} L_i(p,W)) ,$$

we can simplify Li to

$$\begin{aligned}
 L_i(p, W) = & e^4 \int d^4 k d^4 \ell \{ [(p-k)^2 - W^2] \cdot [(p-k-\ell)^2 - W^2] \cdot [(p-\ell)^2 - W^2] \cdot k^2 \cdot \ell^2 \}^{-1} \\
 & \times \{ 4W[(p-k) \cdot (p-k-\ell) + (p-\ell) \cdot (p-k-\ell) + (p-k) \cdot (p-\ell) - 2W^2] \\
 & + \frac{4}{p} [W^2(p(p-k) + p(p-\ell) + p(p-k-\ell)) - 2p(p-k-\ell)(p-k) \cdot (p-\ell)] \} . \quad (28)
 \end{aligned}$$

Because the first two cut diagrams in Fig. [A-2] are equivalent, we have



$$\begin{aligned}
 & = (-i2\pi)^2 e^4 \int d^4 k d^4 \ell \frac{\delta_+((p-k)^2 - W^2) \delta_+(k^2)}{[(p-k-\ell)^2 - W^2] [(p-\ell)^2 - W^2] \ell^2} \\
 & \times \{ 4W[(p-\ell)^2 - 2p \cdot \ell + 3k \cdot \ell] + \frac{1}{p} [4W^2(2p^2 + W^2 - 2p \cdot \ell) - 2(p^2 - W^2)^2 \\
 & + 8(p^2 + W^2)p \cdot \ell - 8(p \cdot \ell)^2 + 8(k \cdot \ell)(p \cdot \ell) - 4(p^2 + W^2)k \cdot \ell] \} . \quad (29)
 \end{aligned}$$

Working out the following integrals with the conditions

$$(p-k)^2 = W^2, \quad k^2 = 0$$

$$\int d^4 \ell \frac{1}{[(p-k-\ell)^2 - W^2] \ell^2} = \frac{i}{(4\pi)^2} \left[\ln \frac{\Lambda^2}{W^2} + 1 \right] ,$$

$$\int d^4 \ell \frac{W^2}{[(p-k-\ell)^2 - W^2] [(p-\ell)^2 - W^2] \ell^2} = - \frac{i}{(4\pi)^2} \frac{W^2}{p^2 - W^2} \left[\ln \left| \frac{W^2}{p^2 - W^2} \right| \ln \frac{p^2}{W^2} - f\left(\frac{p^2}{p^2 - W^2}\right) \right] ,$$

$$\int d^4 \ell \frac{p \cdot \ell}{[(p-k-\ell)^2 - W^2] [(p-\ell)^2 - W^2] \ell^2} = - \frac{i}{2(4\pi)^2} \left[2 + \ln \left| \frac{W^2}{p^2 - W^2} \right| \ln \frac{p^2}{W^2} - f\left(\frac{p^2}{p^2 - W^2}\right) \right] ,$$

$$\begin{aligned}
 \int d^4 \ell \frac{k \cdot \ell}{[(p-k-\ell)^2 - W^2] [(p-\ell)^2 - W^2] \ell^2} = & - \frac{i}{2(4\pi)^2} \left[\ln \left| \frac{W^2}{p^2 - W^2} \right| \left(\ln \frac{p^2}{W^2} - \frac{p^2 - W^2}{p^2} \right) \right. \\
 & \left. - f\left(\frac{p^2}{p^2 - W^2}\right) \right] , \quad (30)
 \end{aligned}$$

$$\begin{aligned}
 \int d^4 \ell \frac{(p \cdot \ell)(k \cdot \ell) - (p \cdot \ell)^2}{p^2 [(p-k-\ell)^2 - W^2] [(p-\ell)^2 - W^2] \ell^2} = & \frac{i}{4(4\pi)^2} \left[- \frac{p^2 + W^2}{2p^2} \ln \frac{\Lambda^2}{W^2} + 2 + \frac{p^2 - W^2}{2p^2} \right. \\
 & \left. + \left(\frac{p^2 - W^2}{p^2} \right)^2 \ln \left| \frac{W^2}{p^2 - W^2} \right| \right] ;
 \end{aligned}$$

and using the well-known result

$$2\pi^2 \int d^4k \delta_+((p-k)^2 - W^2) \delta_+(k^2) = \frac{\pi}{(4\pi)^2} \cdot \frac{p^2 - W^2}{p^2} \theta(p^2 - W^2),$$

we arrive at

$$(29) = i[WX_1(p^2, W^2) + pX_2(p^2, W^2)] \theta(p^2 - W^2) \quad (31)$$

$$\text{with } X_1(p^2, W^2) = -8\pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{p^2 - W^2}{p^2} \left\{ \ln \frac{\Lambda^2}{W^2} + 3 - \frac{p^2 + W^2}{2(p^2 - W^2)} \ln \frac{p^2}{W^2} \cdot \ln \frac{W^2}{p^2 - W^2} \right. \\ \left. + \frac{p^2 + W^2}{2(p^2 - W^2)} f\left(\frac{p^2}{p^2 - W^2}\right) + \frac{3(p^2 - W^2)}{2p^2} \ln \frac{W^2}{p^2 - W^2} \right\},$$

$$\text{and } X_2(p^2, W^2) = -2\pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{p^2 - W^2}{p^2} \left\{ -\frac{p^2 + W^2}{p^2} \left(\ln \frac{\Lambda^2}{W^2} + 1 \right) - 2 + \frac{4W^2}{p^2 - W^2} f\left(\frac{p^2}{p^2 - W^2}\right) \right. \\ \left. - 4 \frac{W^2}{p^2 - W^2} \ln \frac{W^2}{p^2 - W^2} \ln \frac{p^2}{W^2} - 4 \frac{W^2(p^2 - W^2)}{p^4} \ln \frac{W^2}{p^2 - W^2} \right\}.$$

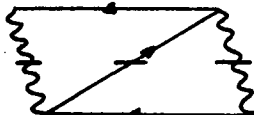
Above, the f -function is defined by

$$f(x) = \int_0^1 dt \frac{\ln(1-t)}{x-t},$$

of which an obvious property is

$$\lim_{x \rightarrow \infty} xf(x) = -1.$$

The third diagram in Fig.[A-2] is



$$= i4\pi^3 e^4 \int d^4k d^4\ell \delta_+((p-k-\ell)^2 - W^2) \delta_+(k^2) \delta_+(\ell^2) \quad (32)$$

$$\times \{ [2W(p^2 - W^2) + 4pW^2] + 4(W-p) \cdot \frac{p^2 - W^2}{p^2} p \cdot (p-k-\ell) \} / \{ [(p-k)^2 - W^2][(p-\ell)^2 - W^2] \}.$$

(32) consists of the following two kinds of integrals, which we cast into a two components column matrix for the sake of convenience,

$$\begin{aligned}
& 4\pi^3 \int \frac{d^4 k d^4 \ell}{d^4 k d^4 \ell} \frac{\delta_+((p-k-\ell)^2 - W^2) \delta_+(k^2) \delta_+(\ell^2)}{[(p-k)^2 - W^2][(p-\ell)^2 - W^2]} \begin{bmatrix} 1 \\ p \cdot (p-k-\ell)/p^2 \end{bmatrix} \\
&= \frac{2\pi}{(4\pi)^4} \cdot \frac{1}{p^2} \int_0^{(p-|W|)^2} dq^2 \frac{\ln \Phi(q^2; p^2, W^2)}{p^2 - W^2 - q^2} \begin{bmatrix} 1 \\ (p^2 + W^2 - q^2)/2p^2 \end{bmatrix} \quad (33) \\
&\times \theta(p^2 - W^2) \quad , \quad \text{with } p = (p^2)^{1/2} .
\end{aligned}$$

In (33), Φ is related to a triangle function

$$\Delta(q^2; p^2, W^2) = [(p^2 + q^2 - W^2)^2 - 4p^2 q^2]^{\frac{1}{2}}$$

through

$$\Phi(q^2; p^2, W^2) = \left| \frac{p^2 - W^2 - q^2 + \Delta(q^2; p^2, W^2)}{p^2 - W^2 - q^2 - \Delta(q^2; p^2, W^2)} \right| .$$

Let us define the functions $Z_1(p^2, W^2)$ and $Z_2(p^2, W^2)$ by

$$\begin{bmatrix} Z_1(p^2, W^2) \\ Z_2(p^2, W^2) \end{bmatrix} = \int_0^{(p-|W|)^2} dq^2 \ln \Phi(q^2; p^2, W^2) \begin{bmatrix} \frac{1}{p^2 - q^2 - W^2} \\ \frac{1}{p^2} \end{bmatrix} \theta(p^2 - W^2) . \quad (34)$$

we can express (32), in terms of the Z 's, as

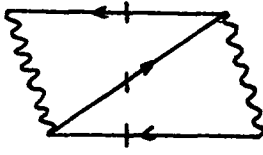
$$(32) = i4\pi \left(\frac{e}{4\pi}\right)^4 \left[\left(\frac{p^2 + W^2}{p^2} W + 2 \frac{W^4}{p^4} \right) Z_1(p^2, W^2) + (W - p \frac{p^2 - W^2}{p^2}) Z_2(p^2, W^2) \right] . \quad (35)$$

The last diagram is the one with two electron and one position lines cut, the evaluation of which is more complicated than the previous ones.

By changing variables to

$$k' = p - k \quad , \quad \ell' = p - \ell$$

it can be expressed as



$$\begin{aligned}
 &= i4\pi^3 e^4 \int \bar{d}^4 k' \bar{d}^4 \ell' \frac{1}{(p-k')^2 (p-\ell')^2} \delta_+(k'^2 - W^2) \\
 &\quad \times \delta_+(\ell'^2 - W^2) \delta_+((p-k'-\ell')^2 - W^2) \\
 &\quad \times \left\{ 4W[2p \cdot (k' + \ell') - \frac{3}{2}(p^2 + W^2)] + \frac{4}{p} [3(p^2 + W^2)p \cdot (k' + \ell') \right. \\
 &\quad \left. - 2(p \cdot (k' + \ell'))^2 - p^2(p^2 + 2W^2)] \right\}. \quad (36)
 \end{aligned}$$

Very long calculations lead us to

$$(36) = i8\pi \left(\frac{e}{4\pi}\right)^4 [WY_1(p^2, W^2) + pY_2(p^2, W^2)], \quad (37)$$

with, for p^2 greater than $9W^2$,

$$\begin{aligned}
 \begin{bmatrix} Y_1(p^2, W^2) \\ Y_2(p^2, W^2) \end{bmatrix} &= \frac{1}{p^2} \int_{4W^2}^{(p-|W|)^2} dq^2 \frac{\ln \Psi(q^2; p^2, W^2)}{p^2 + 3W^2 - q^2} \theta(p^2 - 9W^2) \\
 &\quad \times \left[\begin{aligned} &q^2 - \frac{1}{2}(p^2 + 5W^2) \\ &[\frac{1}{2}(p^2 + 5W^2)q^2 - \frac{1}{2}q^4 - 2W^4 - p^2 W^2] / p^2 \end{aligned} \right], \quad (38)
 \end{aligned}$$

and the $\Psi(q^2; p^2, W^2)$ is defined by

$$\Psi(q^2; p^2, W^2) = \left| \frac{p^2 + 3W^2 - q^2 + (1 - 4W^2/q^2)^{1/2} \Delta(q^2; p^2, W^2)}{p^2 + 3W^2 - q^2 - (1 - 4W^2/q^2)^{1/2} \Delta(q^2; p^2, W^2)} \right|,$$

including the same $\Delta(q^2; p^2, W^2)$ as we encountered before.

In the above calculations, we have written out the explicit locations of the cuts. However we should keep in mind that, for all quantities but Y_s the cuts begin from $p^2 = W^2$, while that for the Y_s begins at $p^2 = 9W^2$.

To evaluate the last, vacuum polarization contribution $\Sigma_Y(p, W)$, we disentangle the algebras first

$$\begin{aligned} \Sigma_Y(p, W) = & 4\left(\frac{e}{4\pi}\right)^2 \left\{ ie^2 \int \bar{d}^4 k \gamma_\mu \frac{K(k^2)}{p-k-W} \gamma^\mu \frac{1}{k^2} \right. \\ & \left. - ie^2 \int \bar{d}^4 k \frac{K(k^2)}{k^4} \left[\frac{(p-W)^2}{(p-k)^2 - W^2} (p+W - \frac{p \cdot k}{p}) - (p-W) \right] \right\}. \end{aligned} \quad (39)$$

Now applying the cutting rules to (39) and noting that $K(0)=0$, we arrive at

$$\begin{aligned} \text{Im} \Sigma_Y(p, W) = & 8\pi^2 e^2 \left(\frac{e}{4\pi}\right)^2 \cdot (p-W)^2 \left[p+W - \frac{p^2 - W^2}{2p} \right] \frac{d}{d\mu^2} K(\mu^2) \Big|_{\mu^2=0} \\ & \times \int \bar{d}^4 k \delta_+(k^2) \delta_+((p-k)^2 - W^2) \\ = & 4\pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{(p^2 - W^2)^3}{2p^2} \frac{d}{d\mu^2} K(\mu^2) \Big|_{\mu^2=0} \cdot \theta(p^2 - W^2). \end{aligned} \quad (40)$$

Recalling (III-61), we obtain the final result for the absorptive part of $\Sigma_Y(p, W)$

$$\text{Im} \Sigma_Y(p, W) = \frac{-2\pi}{15} \left(\frac{e}{4\pi}\right)^4 \cdot \frac{(p^2 - W^2)^3}{p^2} \theta(p^2 - W^2). \quad (41)$$

At this stage we are ready to add all the parts of $\text{Im} \Sigma(p, W)$. For the sake of clarity we write the finite and infinite parts separately.

Let

$$\text{Im} \Sigma(p, W) = \text{Im} \Sigma_f(p, W) + \text{Im} \Sigma_{\text{inf}}(p, W), \quad (42)$$

then we have

$$\text{Im} \Sigma_{\text{inf}}(p, W) = -3 \left(\frac{e}{4\pi}\right)^2 \cdot \left[\ln \frac{\Lambda^2}{W^2} + 1 \right] \cdot \text{Im} \Sigma^{(0)}(p, W), \quad (43)$$

and

$$\begin{aligned} \text{Im} \Sigma_f(p, W) = & \pi \left(\frac{e}{4\pi}\right)^2 \cdot \frac{p^2 - W^2}{p^2} \left[-(a+3)W + a p \frac{p^2 + W^2}{p^2} \right] \\ & + \pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{p^2 - W^2}{p^2} \left[-(a+3)W + a p \frac{p^2 + W^2}{p^2} \right] \cdot \frac{p+W}{p^2} \\ & \times \left\{ 2(4W-p) \frac{p^2}{p^2 - W^2} \ln \frac{p^2}{W^2} + 2(4W-p) \frac{p^2 + W^2}{p^2} \ln \frac{W^2}{p^2 - W^2} + p \right\} \end{aligned}$$

$$\begin{aligned}
& - \pi \left(\frac{e}{4\pi} \right)^4 \cdot \frac{p^2 - W^2}{p^2} \left\{ -(a+3) \left[\left(\frac{4W^2}{p^2 - W^2} \ln \frac{p^2}{W^2} - 3 \right) W + p \frac{W^2}{p^2} \right] \right. \\
& + a \left[\left(\frac{p^2 - W^2}{p^2} + \frac{4p^2}{p^2 - W^2} \ln \frac{p^2}{W^2} \right) W + p \left(\frac{9W^2 - 7p^2}{2p^2} + \frac{4W^4 - p^4}{p^2(p^2 - W^2)} \ln \frac{p^2}{W^2} \right) \right] \} \\
& - \pi \left(\frac{e}{4\pi} \right)^4 \cdot \frac{p^2 - W^2}{p^2} \left[-4W + p \frac{p^2 + W^2}{p^2} \right] \\
& - \pi \left(\frac{e}{4\pi} \right)^4 \cdot (1-a) \frac{p^2 - W^2}{p^2} \left\{ W \left[\frac{3p^2 - 2W^2}{p^2} + 2 \frac{W^4 + p^2 W^2 - 4p^4}{p^4} \ln \frac{W^2}{p^2 - W^2} \right. \right. \\
& \quad \left. \left. - 3 \frac{W^2}{p^2 - W^2} \ln \frac{p^2}{W^2} \right] \right. \\
& + p \left[\frac{3W^2 - p^2}{2p^2} + 2 \frac{p^2 W^2 + p^4 - 4W^4}{p^4} \ln \frac{W^2}{p^2 - W^2} \right] \} \\
& - \pi \left(\frac{e}{4\pi} \right)^4 \cdot \frac{p^2 - W^2}{p^2} \left\{ 8W \left[2 + \frac{p^2 + W^2}{2(p^2 - W^2)} f \left(\frac{p^2}{p^2 - W^2} \right) + \frac{3(p^2 - W^2)}{2p^2} \ln \frac{W^2}{p^2 - W^2} \right. \right. \\
& \quad \left. \left. - \frac{p^2 + W^2}{2(p^2 - W^2)} \ln \frac{p^2}{W^2} \ln \frac{W^2}{p^2 - W^2} \right] \right. \\
& + 2p \left[-2 + \frac{4W^2}{p^2 - W^2} f \left(\frac{p^2}{p^2 - W^2} \right) - 4 \frac{W^2}{p^2 - W^2} \ln \frac{W^2}{p^2 - W^2} \ln \frac{p^2}{W^2} \right. \\
& \quad \left. \left. - 4 \frac{W^2(p^2 - W^2)}{p^4} \ln \frac{W^2}{p^2 - W^2} \right] \right\} \\
& + 4\pi \left(\frac{e}{4\pi} \right)^4 \cdot \left[\left(\frac{p^2 + W^2}{p^2} W + 2 \frac{W^4}{p^4} p \right) Z_1(p^2, W^2) + \left(W - p \frac{p^2 - W^2}{p^2} \right) Z_2(p^2, W^2) \right] \\
& + 8\pi \left(\frac{e}{4\pi} \right)^4 \cdot [WY_1(p^2, W^2) + pY_2(p^2, W^2)] \\
& + 4\pi \left(\frac{e}{4\pi} \right)^4 \cdot \frac{p^2 - W^2}{p^2} \left[-(a+3)W + ap \frac{p^2 + W^2}{p^2} \right] \ln \frac{p^2}{W^2} \\
& - \pi \frac{2}{15} \left(\frac{e}{4\pi} \right)^4 \cdot \frac{(p^2 - W^2)^3}{p^4 m^2} p,
\end{aligned} \tag{44}$$

with the cuts understood.

APPENDIX B. THE EVALUATION OF $\Sigma(p^2, w^2)$

This appendix is devoted to the evaluation of the mass operator $\Sigma(p^2, w^2)$ defined by (IV-43). We decompose it into four parts

$$\Sigma(p^2, w^2) = \Sigma^{(0)}(p^2, w^2) + \Sigma_Y(p^2, w^2) + \Sigma_L(p^2, w^2) + \Sigma_T(p^2, w^2) \quad , \quad (1)$$

with $\Sigma^{(0)}(p^2, w^2)$ given in (IV-36) and

$$\Sigma_Y(p^2, w^2) = -ie^2 \int \bar{d}^4 q \frac{(2p-q)_\mu (2p-q)_\nu}{(p-q)^2 - w^2} \cdot \frac{K(q^2)}{q^2} (-g^{\mu\nu} + q^\mu q^\nu / q^2) \quad , \quad (2)$$

$$\begin{aligned} \Sigma_L(p^2, w^2) = e^4 \int \bar{d}^4 t \bar{d}^4 t' g_{\mu\nu} \frac{D^{\mu\alpha}_{(0)}(t') D^{\nu\beta}_{(0)}(t)}{(p-t-t')^2 - w^2} \\ \times \left\{ 2g_{\alpha\beta} - \frac{[2(p-t)-t']_\alpha (2p-t)_\beta}{(p-t)^2 - w^2} - \frac{[2(p-t')-t]_\beta (2p-t')_\alpha}{(p-t')^2 - w^2} \right\} \quad (3) \end{aligned}$$

$$\begin{aligned} \Sigma_T(p^2, w^2) = e^4 \int \bar{d}^4 q \bar{d}^4 t \frac{(2p-q)_\mu D^{\mu\nu}_{(0)}(q) D^{\sigma\pi}_{(0)}(t)}{(p-q)^2 - w^2} \\ \times \left\{ \frac{[(2p-q)+2t]_\nu (2p+t)_\sigma [2(p-q)+t]_\pi}{[(p+t)^2 - w^2][(p-q+t)^2 - w^2]} \right. \\ - 2g_{\nu\sigma} \left[\frac{2(p-q)_\pi + t_\pi}{(p-q+t)^2 - w^2} + \frac{(2p+t)_\pi}{(p+t)^2 - w^2} \right] \\ + \frac{(2p-q)_\nu}{(p-q)^2 - p^2} \left[\frac{(2(p-q)+t)_\sigma (2(p-q)+t)_\pi}{(p-q+t)^2 - w^2} \right. \\ \left. \left. - \frac{(2p+t)_\sigma (2p+t)_\pi}{(p+t)^2 - w^2} \right] \right\} \quad (4) \end{aligned}$$

where $\Sigma^{(0)}$ is the usual lowest order mass correction; Σ_Y is the contribution from photon vacuum polarization; Σ_L and Σ_T , consisting of the e^4 order terms of Σ , are the lowest order $G_{\mu\nu}$ and transverse G_μ contributions respectively.

We first evaluate the simplest piece $\Sigma^{(0)}$. Its absorptive part is so simple to work out that we will not give any explanation but merely

state the end result

$$\text{Im}\Sigma^{(0)}(p^2, W^2) = \pi \left(\frac{e}{4\pi}\right)^4 \cdot (a-3) \cdot \frac{p^4 - W^4}{p^2} \theta(p^2 - W^2). \quad (5)$$

To calculate $\Sigma^{(0)}(W^2, W^2)$, we decompose (IV-36) into a gauge independent and a gauge dependent part

$$\Sigma_i^{(0)}(p^2, W^2) = ie^2 \int \bar{d}^4 q \frac{(2p-q)^2}{[(p-q)^2 - W^2]q^2} \quad (6)$$

$$\Sigma_g^{(0)}(p^2, W^2) = ie^2(1-a) \int \bar{d}^4 q \frac{[q \cdot (2p-q)]^2}{[(p-q)^2 - W^2]q^4} \quad (7)$$

with

$$\Sigma^{(0)}(p^2, W^2) = \Sigma_i^{(0)}(p^2, W^2) - \Sigma_g^{(0)}(p^2, W^2) .$$

It is very easy to show $\Sigma_g^{(0)}(W^2, W^2) = 0$, therefore we are only interested in (6). Under dimensional regularization we can simplify the $p^2=W^2$ version of (6) to

$$\Sigma_i^{(0)}(W^2, W^2) = ie^2 \int \bar{d}^{2\ell} q (M^2)^{2-\ell} \frac{4W^2 - 2p \cdot q}{[(p-q)^2 - W^2]q^2} \Big|_{p^2=W^2} \quad (8)$$

by noting that

$$\int \bar{d}^{2\ell} q \frac{1}{q^2} = 0 .$$

Working out the following integrals, under the condition $p^2=W^2$,

$$ie^2 (M^2)^{2-\ell} \int \bar{d}^{2\ell} q \frac{4W^2}{[(p-q)^2 - W^2]q^2} = -4 \left(\frac{e}{4\pi}\right)^2 W^2 [\Gamma(2-\ell) \left(\frac{M^2}{W^2}\right)^{2-\ell} + 2]$$

$$ie^2 (M^2)^{2-\ell} \int \bar{d}^{2\ell} q \frac{2p \cdot q}{[(p-q)^2 - W^2]q^2} = -\left(\frac{e}{4\pi}\right)^2 W^2 [\Gamma(2-\ell) \left(\frac{M^2}{W^2}\right)^{2-\ell} + 1]$$

and using them in (8), we obtain

$$\Sigma^{(0)}(W^2, W^2) = -\left(\frac{e}{4\pi}\right)^2 \cdot W^2 [3\Gamma(2-\ell) \left(\frac{M^2}{W^2}\right)^{2-\ell} + 7] . \quad (9)$$

Now we turn to the evaluation of Σ_L . Let the first integral of (3) be denoted by $I_1(p^2, w^2)$. Carrying out the contractions, we have

$$I_1(p^2, w^2) = 2e^4 \int \bar{d}^4 t \bar{d}^4 t' [-2(1+a) + \frac{(1-a)^2 \cdot (t \cdot t')^2}{t^2 t'^2}] \{t'^2 \cdot t^2 \cdot [(p-t-t')^2 - w^2]\}^{-1}. \quad (10)$$

With the following integrals [83],

$$\begin{aligned} & \frac{(2\pi)^3}{2} \int \bar{d}^4 t \bar{d}^4 t' \delta_+(t) \delta_+(t') \delta_+[(p-t-t')^2 - w^2] \\ &= \pi \left(\frac{e}{4\pi}\right)^2 \cdot \int_{w^2}^{p^2} dS \frac{1}{p^2 S} (p^2 - S)(w^2 - S) \theta(p^2 - w^2), \end{aligned} \quad (11)$$

$$\begin{aligned} & \frac{(2\pi)^3}{2} \int \bar{d}^4 t \bar{d}^4 t' (t \cdot t')^2 \cdot \delta'_+(t^2) \delta'_+(t'^2) \delta_+[(p-t-t')^2 - w^2] \\ &= -\frac{5}{4} \pi \left(\frac{e}{4\pi}\right)^4 \int_{w^2}^{p^2} dS \frac{1}{p^2 S} (p^2 - S)(w^2 - S) \theta(p^2 - w^2), \end{aligned} \quad (12)$$

We can easily work out the imaginary part of $I_1(p^2, w^2)$ using Cutkosky-Nakanishi [82] cutting-rules. The result is

$$\text{Im} I_1(p^2, w^2) = -4\pi \left(\frac{e}{4\pi}\right)^2 \cdot \int_{w^2}^{p^2} \frac{dS}{p^2 S} [(1+a) + \frac{5}{8} (1-a)^2] (p^2 - S)(w^2 - S) \theta(p^2 - w^2). \quad (13)$$

The evaluation of the second integral of (3) is a bit more complicated.

We denote it by $\frac{1}{2} I_2(p^2, w^2)$, then rewrite it as follows

$$\frac{1}{2} I_2(p^2, w^2) = ie^4 \int \bar{d}^4 t \frac{(2p-t)_\mu}{(p-t)^2 - w^2} D_{(0)}^{\mu\nu}(t) \cdot i \int \bar{d}^4 t' \frac{[2(p-t)-t']_\lambda D_{(0)}^\lambda{}_\nu(t')}{(p-t-t')^2 - w^2}. \quad (14)$$

Let

$$Z(p^2, w^2) p_\nu = i \int \bar{d}^4 t \frac{(2p-t)_\mu D_{(0)}^\mu{}_\nu(t)}{(p-t)^2 - w^2}. \quad (15)$$

It is quite easy to show that ($\ell \rightarrow 2$)

$$Z(p^2, W^2) p_\nu = p_\nu i(M^2)^{2-\ell} \int d^{2\ell} t' \int_0^1 d\alpha \left\{ \frac{-(2-\alpha)}{[t'^2 - \alpha(\alpha-1)p^2 - \alpha W^2]^2} \right. \\ \left. + (1-\alpha) \frac{(p^2 - W^2) \alpha(1-\alpha) \Gamma(3)}{[t'^2 - \alpha(\alpha-1)p^2 - \alpha W^2]^3} \right\} \quad (16)$$

under dimensional regularization. Working out the following integrals

$$\text{Im}\{-i(M^2)^{2-\ell} \int d^{2\ell} t' \int_0^1 d\alpha \frac{2-\alpha}{[t'^2 - \alpha(\alpha-1)p^2 - \alpha W^2]^2}\} \\ = \frac{\pi}{(4\pi)^2} \cdot \frac{p^2 - W^2}{p^2} \cdot \frac{3p^2 + W^2}{2p^2} \theta(p^2 - W^2) \quad (17)$$

$$\text{Im}\{i(M^2)^{2-\ell} \int d^{2\ell} t' \int_0^1 d\alpha \frac{\alpha(1-\alpha) \Gamma(3)}{[t'^2 - \alpha(\alpha-1)p^2 - \alpha W^2]^3}\} = \frac{\pi}{(4\pi)^2} \cdot \frac{W^2}{p^4} \theta(p^2 - W^2)$$

then using them in (16), we obtain

$$\text{Im}Z(p^2, W^2) = \frac{1}{2(4\pi)^2} \cdot \frac{p^2 - W^2}{p^4} \cdot [3p^2 + (3-2a)W^2] \theta(p^2 - W^2) \quad (18)$$

Setting $p^2 = W^2$, the second term of (16) vanishes, and the first part simply gives

$$Z(W^2, W^2) = \frac{1}{2(4\pi)^2} [3\Gamma(2-\ell) \left[\frac{M^2}{W^2} \right]^{2-\ell} + 7] \quad (19)$$

From (16) we can see that $Z(p^2, W^2)$ is logarithmically divergent, and therefore can be represented by a once-subtracted dispersion relation

$$Z(p^2, W^2) = Z(W^2, W^2) + \frac{p^2 - W^2}{\pi} \int dS \frac{\text{Im}Z(S, W^2)}{(S - W^2)(S - p^2 - i0^+)} \quad (20)$$

Inserting this into (14) and analytically continuing the dimension of the space-time to 21, leads us to

$$\begin{aligned}
I_2(p^2, w^2) = & 2e^4 i \int \bar{d}^{2\ell} t (M^2)^{2-\ell} \frac{Z(W^2, w^2)}{t^2 [(p-t)^2 - w^2]} [-p \cdot (2p-t) + a(p^2 - w^2) \\
& + (1-a) \frac{(p^2 - w^2)p \cdot t}{t^2}] \\
& - 2e^4 i \frac{1}{\pi} \int dS \frac{\text{Im} Z(S, w^2)}{S - w^2} \int \bar{d}^{2\ell} t \frac{(M^2)^{2-\ell}}{t^2 [(p-t)^2 - S]} [-p \cdot (2p-t) \\
& + a(p^2 - S) + (1-a) \frac{(p^2 - S)p \cdot t}{t^2}] . \quad (21)
\end{aligned}$$

Equation (21) consists of three kinds of integrals. Their absorptive parts are given here in three-vector notation for the sake of compactness.

$$\begin{aligned}
& \text{Im} \int \bar{d}^{2\ell} t \frac{i}{t^2 [(p-t)^2 - w^2]} \begin{bmatrix} p \cdot (2p-t)/p^2 \\ 1 \\ p \cdot t/t^2 \end{bmatrix} \\
& = \frac{\pi}{(4\pi)^2} \theta(p^2 - w^2) \begin{bmatrix} -(p^2 - w^2)(3p^2 + w^2)/2p^4 \\ -(p^2 - w^2)/p^2 \\ w^2/p^2 \end{bmatrix} . \quad (22)
\end{aligned}$$

Taking the imaginary part of (21) and substituting (22) into the resultant equation we arrive, after some tedious manipulations, at the expression

$$\begin{aligned}
\text{Im} I_2(p^2, w^2) = & -(4\pi)^2 \cdot \left(\frac{e}{4\pi}\right)^4 \cdot Z(W^2, w^2) (p^2 - w^2) [(2a-3)p^2 - 3w^2] \theta(p^2 - w^2)/p^2 \\
& + (4\pi)^2 \cdot \left(\frac{e}{4\pi}\right)^4 \int dS \frac{\text{Im} Z(S, w^2)}{(S - w^2)^2 p^2} (p^2 - S) [(2a-3)p^2 - 3S] \theta(p^2 - w^2) . \quad (23)
\end{aligned}$$

With the use of (18) and (19), we can obtain a very neat expression for the imaginary part of I_2

$$\begin{aligned}
\text{Im} I_2(p^2, w^2) = & -\pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{1}{p^2} \int_{w^2}^{p^2} dS \left\{ \frac{p^2 - S}{2S^2} [3S + (3-2a)w^2] \cdot [3S + (3-2a)p^2] \right. \\
& \left. - \frac{1}{2} [3w^2 + (3-2a)p^2] [3\Gamma(2-\ell) \left(\frac{M^2}{w^2}\right)^{2-\ell} + 7] \right\} \cdot \theta(p^2 - w^2) . \quad (24)
\end{aligned}$$

The third term of Σ_L is also $\frac{1}{2} I_2$, so we do not need to calculate it again.

Before starting the evaluation of Σ_T , we consider some of its general properties. Because it is the transverse contribution, the part containing the longitudinal piece of $D_{(0)}^{\mu\nu}(q)$ identically vanishes. Therefore, we can replace $D_{(0)}^{\mu\nu}(q)$ by

$$D_{(0)F}^{\mu\nu}(q) = -g^{\mu\nu}/q^2 \quad (25)$$

in (4). Now Σ_T linearly depends on the gauge parameter a , and we can decompose it into

$$\Sigma_T = \Sigma_T^i + (1-a)\Sigma_T^g \quad (26)$$

with Σ_T^i gauge independent. Σ_T^i was worked out by Parker [54] using Cutkosky-Nakanishi cutting-rules and a dimensional regularization scheme. We will not repeat the mathematical details here but quote the result with certain minor modifications to make it consistent with our previous notation,

$$\begin{aligned} \text{Im}\Sigma_T^i(p^2, W^2) = & \pi \left(\frac{e}{4\pi}\right)^4 \frac{p^2 - W^2}{4p^2} \left\{ 2(3p^2 + W^2) [3\Gamma(2-\ell) \left[\frac{M^2}{W^2}\right]^{2-\ell} + 7] \right. \\ & - (75p^2 + 143W^2) - \frac{2p^2(5p^2 + 6W^2)}{p^2 - W^2} \ln \frac{p^2}{W^2} \\ & - 4 \frac{(p^2 + 3W^2)^2}{p^2 - W^2} \left[2 \ln \frac{p^2}{W^2} \cdot \ln \frac{p^2 - W^2}{W^2} + 3f\left(\frac{p^2}{W^2}\right) \right] \\ & \left. + 8 \frac{p^4 + 14p^2W^2 + W^4}{p^4} \ln \frac{p^2 - W^2}{W^2} \right\} \alpha(p^2 - W^2) + \theta(p^2 - 9W^2) I(p^2, W^2), \end{aligned} \quad (27)$$

with $f(x)$ the Spence function [85], defined by

$$f(x) = \int_1^x dt \frac{\ln t}{1-t}.$$

As the integrand is regular at $t=1$ because

$$\lim_{t \rightarrow 1} \frac{\ln t}{t-1} = 1$$

we can easily obtain

$$\lim_{x \rightarrow 1} \frac{f(x)}{x-1} = -1$$

$I(p^2, w^2)$ is a very complicated function which can only be expressed as an integral

$$I(p^2, w^2) = 2\pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{1}{p^2} \cdot \int_0^{(p-w)^2} d\ell^2 \left\{ 2pR(p^2, w^2, \ell^2) + \frac{(2p^2 + 2w^2 - \ell^2)(3p^2 + w^2 - 2\ell^2)}{\ell^2 + w^2 - p^2} \ln \left| \frac{4pR(p^2, w^2, \ell^2) + (\ell^2 + w^2 - p^2)}{4pR(p^2, w^2, \ell^2) - (\ell^2 + w^2 - p^2)} \right| \right\} \quad (28)$$

Here $p = \sqrt{p^2}$ is a scalar, and $R(p^2, w^2, \ell^2)$ is a function related to the triangle function

$$\Delta(p^2, w^2, \ell^2) = [(p^2 + w^2 - \ell^2)^2 - 4p^2 w^2]^{\frac{1}{2}} \quad (29)$$

by the expression

$$4pR(p^2, w^2, \ell^2) = \{ [2(p^2 - w^2) - \ell^2] [2(p^2 + w^2) - \ell^2]^{-1} \}^{\frac{1}{2}} \Delta(p^2, w^2, \ell^2) \quad (30)$$

To estimate Σ_T^g , we first write out the photon propagator explicitly in (4) and pick out the proper part according to the definition (26).

Carrying out all the contractions, we end up with

$$\Sigma_T^g(p^2, w^2) = -2e^4 \int \bar{d}^4 q \bar{d}^4 t \frac{p^2 - w^2}{[(p+t)^2 - w^2][(p+t-q)^2 - w^2] t^4 q^2} \times \{ 2p^2 + 2p \cdot t - p \cdot q + \frac{[(p+t)^2 - (p-q+t)^2] \cdot (2p-q) \cdot p}{(p-q)^2 - p^2} \} \quad (31)$$

Up to this point it has not been necessary to leave 4-dimensional Minkowski space since in deriving (31) there has been no reference to dimension. Define

$$J(p^2, w^2) = i \int \bar{d}^4 q \frac{1}{q^2 [(p-q)^2 - w^2]} \quad (32)$$

$$L(p^2, w^2) = i \int \bar{d}^4 t \frac{1}{t^4 [(p+t)^2 - w^2]} \quad (33)$$

$$p \cdot (p+t) K((p+t)^2, w^2) = i \int \bar{d}^4 q \frac{p \cdot q}{q^2 [(p+t-q)^2 - w^2]} \quad (34)$$

we discover that

$$\text{Im} J(p^2, w^2) = - \frac{\pi}{(4\pi)^2} \frac{p^2 - w^2}{p^2} \theta(p^2 - w^2) \quad (35)$$

$$\text{Im} L(p^2, w^2) = + \frac{\pi}{(4\pi)^2} \cdot \frac{p^2 + w^2}{p^2 (p^2 - w^2)} \theta(p^2 - w^2) \quad (36)$$

$$\text{Im} K(p^2, w^2) = - \frac{\pi}{2(4\pi)^2} \left(\frac{p^2 - w^2}{p^2} \right)^2 \cdot \theta(p^2 - w^2) \quad (37)$$

and with dimensional regularization,

$$J(w^2, w^2) = - \frac{1}{(4\pi)^2} [\Gamma(2-\ell) \left[\frac{M^2}{w^2} \right]^{2-\ell} + 2] \quad (38)$$

$$K(w^2, w^2) = - \frac{1}{2(4\pi)^2} [\Gamma(2-\ell) \left[\frac{M^2}{w^2} \right]^{2-\ell} + 1] \quad (39)$$

These results will be used later. By inserting (32)-(34) into (31), it acquires the form

$$\begin{aligned} \Sigma_T^g(p^2, w^2) = i2e^4 (p^2 - w^2) \{ & \int \bar{d}^4 t \frac{p \cdot (p+t)}{[(p+t)^2 - w^2] t^4} [2J((p+t)^2, w^2) - K((p+t)^2, w^2)] \\ & + \int \bar{d}^4 q \frac{(2p-q) \cdot p}{q^2 [(p-q)^2 - p^2]} [L((p-q)^2, w^2) - L(p^2, w^2)] \} . \end{aligned} \quad (40)$$

Noting that we can express $L(p^2, w^2)$ via the dispersion relation

$$L(p^2, w^2) = \frac{1}{\pi} \int dS \frac{\text{Im} L(S, w^2)}{S - p^2 - i0^+} \quad (41)$$

while for both J and K we have the once-subtracted dispersion-relation

$$F(p^2, W^2) = F(W^2, W^2) + \frac{p^2 - W^2}{\pi} \int ds \frac{\text{Im} F(s, W^2)}{(s - W^2)(s - p^2 - i0^+)} \quad (42)$$

we can rewrite (40) as

$$\begin{aligned} \Sigma_T^g(p^2, W^2) = & i2e^4(p^2 - W^2) \left\{ \int \bar{d}^4 t \frac{p \cdot (p+t)}{[(p+t)^2 - W^2 + i0^+](t^2 + i0^+)^2} [2J(W^2, W^2) \right. \\ & \left. - K(W^2, W^2)] \right. \\ & - \frac{1}{\pi} \int ds \frac{\text{Im}[2J(s, W^2) - K(s, W^2)]}{s - W^2} \int \bar{d}^4 t \frac{p \cdot (p+t)}{[(p+t)^2 - s + i0^+](t^2 + i0^+)^2} \\ & \left. + \frac{1}{\pi} \int ds \frac{\text{Im} L(s, W^2)}{p^2 - s + i0^+} \int \bar{d}^4 q \frac{(2p-q) \cdot p}{[(p-q)^2 - s + i0^+](q^2 + i0^+)} \right\} \quad (43) \end{aligned}$$

To avoid confusion we have made explicit the $i0^+$ in the dominators.

Noting that

$$\begin{aligned} \text{Im} \left\{ i \int \bar{d}^4 t \frac{p \cdot (p+t)}{t^4 [(p+t)^2 - W^2]} \right\} &= p^2 \text{Im} L(p^2, W^2) + \text{Im} \left\{ i \int \bar{d}^4 t \frac{p \cdot t}{t^4 [(p+t)^2 - W^2]} \right\} \\ &= p^2 \text{Im} L(p^2, W^2) - \frac{\pi}{(4\pi)^2} \cdot \frac{W^2}{p^2} \theta(p^2 - W^2) \quad (44) \end{aligned}$$

$$\text{and} \quad i \int \bar{d}^4 q \frac{(2p-q) \cdot p}{q^2 [(p-q)^2 - W^2]} = p^2 [2J(p^2, W^2) - K(p^2, W^2)] \quad (45)$$

and taking the imaginary part of (43), we arrive at

$$\begin{aligned} \text{Im} \Sigma_T^g(p^2, W^2) = & 2e^4(p^2 - W^2) \left\{ [2J(W^2, W^2) - K(W^2, W^2)] [p^2 \text{Im} L(p^2, W^2) \right. \\ & \left. - \frac{\pi}{(4\pi)^2} \frac{W^2}{p^2} \theta(p^2 - W^2)] \right. \\ & - \frac{1}{\pi} \int ds \frac{\text{Im}[2J(s, W^2) - K(s, W^2)]}{s - W^2} [p^2 \text{Im} L(p^2, s) - \frac{\pi}{(4\pi)^2} \frac{s}{p^2} \theta(p^2 - s)] \\ & + \frac{1}{\pi} \int ds \frac{\text{Im} L(s, W^2)}{p^2 - s} \cdot p^2 \text{Im} [2J(p^2, s) - K(p^2, s)] \\ & \left. - \text{Im} L(p^2, W^2) p^2 [2J(p^2, p^2) - K(p^2, p^2)] \right\} \quad (46) \end{aligned}$$

Inserting (35)-(39) into (46), we can cast it in the form

$$\begin{aligned} \text{Im} \Sigma_T^g(p^2, W^2) = & \pi \left(\frac{e}{4\pi} \right)^4 \cdot (p^2 - W^2) \left\{ \frac{W^2}{p^2} \left[3\Gamma(2-\ell) \left[\frac{M^2}{W^2} \right]^{2-\ell} + 7 \right] \right. \\ & \left. - \frac{3(p^2 + W^2)}{p^2 - W^2} \ln \frac{p^2}{W^2} + \left[\frac{6p^2 + W^2}{p^2} \ln \frac{p^2}{W^2} - 3 \frac{p^2 - W^2}{p^2} \right] \right\} \theta(p^2 - W^2). \end{aligned} \quad (47)$$

Now the only part which remains to be evaluated is Σ_γ . Σ_γ is connected with the vacuum polarization through the relation

$$K(q^2) = -\pi^{(0)}(q^2) + \pi^{(0)}(0) \quad (48)$$

with $\pi^{(0)}(q^2)$ defined by

$$\pi_{\mu\nu}^{(0)}(q) = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \pi^{(0)}(q^2) \quad (49)$$

where $\pi_{\mu\nu}^{(0)}(q)$ is the lowest order vacuum polarization tensor in perturbation theory, given by (IV-62). To evaluate (48), we first contract out the indices of (IV-62), and then dimensionally regularize it, to obtain

$$\pi_\mu^{(0)\mu}(q) = \left(\frac{e}{4\pi} \right)^2 \cdot q^2 \left\{ -\Gamma(2-\ell) \left[\frac{M^2}{m^2} \right]^{2-\ell} + (4m^2 - q^2) \int_0^1 d\alpha \frac{(2\alpha-1)\alpha}{\alpha(\alpha-1)q^2 + m^2} \right\}. \quad (50)$$

Therefore

$$K(q^2) = -\frac{1}{3} \left(\frac{e}{4\pi} \right)^2 q^2 \int_0^1 d\alpha \frac{(2\alpha-1)^3 \cdot \alpha}{\alpha(\alpha-1)q^2 + m^2}. \quad (51)$$

Equation (51) can be further simplified to

$$K(q^2) = \frac{1}{3} \left(\frac{e}{4\pi} \right)^2 \cdot q^2 \int_{4m^2}^{\infty} dM^2 \frac{(1 - \frac{4m^2}{M^2})^{3/2}}{M^2(q^2 - M^2)}. \quad (52)$$

We differentiate (52) with respect to q^2 , then set q^2 to zero, and in so doing, arrive at

$$K'(0) = -\frac{1}{30} \cdot \left(\frac{e}{4\pi}\right)^2 \cdot \frac{1}{m^2} . \quad (54)$$

Next consider (2). Some simple calculations lead us to

$$\Sigma_Y(p^2, W^2) = -ie^2 \int d^4q \left\{ -\frac{(2p-q)^2 K(q^2)}{[(p-q)^2 - W^2] q^2} + \frac{K(q^2)[(p^2 - W^2) + W^2 - (p-q)^2]^2}{[(p-q)^2 - W^2] q^4} \right\} . \quad (55)$$

Noting that in the vanishing q^2 limit $K(q^2)$ behaves like $\frac{q^2}{m^2}$, we can see that only the second term of (55) will contribute to the imaginary part of Σ_Y . Upon applying the cutting-rules to it and using (54) we obtain

$$\begin{aligned} \text{Im } \Sigma_Y(p^2, W^2) &= 2\pi^2 e^2 K'(0) (p^2 - m^2)^2 \int d^4q \delta_+(q^2) \delta_+[(p-q)^2 - W^2] \\ &= -\frac{1}{30} \pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{(p^2 - W^2)^3}{p^2 m^2} \theta(p^2 - W^2) . \end{aligned} \quad (56)$$

In the infrared limit, $p^2 \rightarrow m^2$, $W^2 \rightarrow m^2$,

$$\text{Im } \Sigma_Y(p^2, W^2) \rightarrow \alpha (p^2 - W^2)^3 . \quad (57)$$

Therefore in this region the vacuum polarization does not have much effect on the charged particle propagator, as the imaginary mass correction $\text{Im } \Sigma$ contains terms behaving like $O(p^2 - W^2)$. Adding all the different terms together, we arrive at the final result for the imaginary part of $\Sigma(p^2, W^2)$

$$\begin{aligned} \text{Im } \Sigma(p^2, W^2) &= \pi \left(\frac{e}{4\pi}\right)^2 \cdot (a-3) \frac{p^4 - W^4}{p^2} \\ &\quad - \pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{1}{p^2} \int_{W^2}^{p^2} dS \left\{ \frac{4}{S} [(1+a) + \frac{5}{8} (1-a)^2] (p^2 - S)(W^2 - S) \right. \\ &\quad \left. + \frac{p^2 - S}{2S^2} [3S + (3-2a)p^2][3S + (3-2a)W^2] \right\} \\ &\quad - \pi \left(\frac{e}{4\pi}\right)^4 \cdot \frac{p^2 - W^2}{4p^2} \left\{ (75p^2 + 143W^2) + \frac{2p^2(5p^2 + 6W^2)}{p^2 - W^2} \ln \frac{p^2}{W^2} \right. \\ &\quad \left. + 4 \frac{(p^2 + 3W^2)^2}{p^2 - W^2} \left[3f\left(\frac{p^2}{W^2}\right) - 21 \ln \frac{p^2}{W^2} \ln \frac{W^2}{p^2 - W^2} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + 8 \frac{p^4 + 14p^2w^2 + w^4}{p^2} \ln \frac{w^2}{p^2 - w^2} \} + \theta(p^2 - 9w^2) I(p^2, w^2) \\
& - \pi \left(\frac{e}{4\pi}\right)^4 (1-a) \cdot (p^2 - w^2) \left\{ \frac{3(p^2 + w^2)}{p^2 - w^2} \ln \frac{p^2}{w^2} \right. \\
& \quad \left. + \left[3 \frac{p^2 - w^2}{p^2} - 6 \frac{p^2 + w^2}{p^2} \ln \frac{p^2}{w^2} \right] \right\} \\
& - \frac{\pi}{30} \left(\frac{e}{4\pi}\right)^4 \cdot \frac{(p^2 - w^2)^3}{m^2 p^2} + \pi \left(\frac{e}{4\pi}\right)^4 (3-a) \frac{p^4 - w^4}{p^2} \left[3\Gamma(2-\ell) \left[\frac{M^2}{w^2} \right]^{2-\ell} + 7 \right].
\end{aligned}
\tag{58}$$

Above we have omitted the $\theta(p^2 - w^2)$ function attached to all the terms but the 1.

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